



Bifurcation of Solution in Singularly Perturbed Ordinary Differential Equations and Differential Algebraic Equations by Extended System Reduction

A THESIS SUBMITTED TO THE COUNCIL OF THE COLLEGE OF
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CERTIFICATE

It is certified that the work incorporated in the thesis entitled (**Bifurcation of Solution in Singularly Perturbed Ordinary Differential Equations and Differential Algebraic Equations by Extended System Reduction.**) submitted by (**Ayat Salam Al-Jabry**). It comprises the results of independent and original investigation carried out by the candidate under my supervision. The material that has been obtained (and used) from other sources have been duly acknowledged in the thesis.

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Dedication

To My Parents, My Brothers and My Husband

Acknowledgement

First, all the praise and thanks to **ALLAH** Almighty for his mercy and blessings to enable me to continue this work and to complete the requirements of this thesis. My Thanks and my gratitude to the professor and supervisor **Dr.Kamal Hamid AL-Mousawi** for his continuous help in completing this work. And the heart stem thanks to those who support me in my knowledge and learning, I thank my parents for their patience and support in the stages of my long study and thank my beloved husband for encouraging me to do this work. I would like to express my thanks to my teachers and friends for their support throughout the time.

Ayat salam

List Of Symbols

Table 1: Table of Notation.

The following notation will be used throughout the thesis.

Notation	Comment
$\dot{x}$	Derivative of variable x with respect to the time t .
$\acute{x}$	Derivative of variable x with respect to the time τ .
x_*	The equilibrium point.
$\mathcal{C}$	Is a set of all equilibrium points.
R	The set of real number.
R^n	The set of real value vectors.
f_x	Partial derivative of f with respect to x .
g_z	Partial derivative of g with respect to z .
ϵ	A perturbation parameter.
μ	A bifurcation parameter.
G	Extended system reduction.
G_y	The Jacobian matrix.
$rank$	The rank of a matrix.
$\mathcal{S} = F^{-1}(0)$	Fold curve of F .
$DAEs$	Differential Algebraic Equation system.
$ODEs$	Ordinary Differential Equation system.
L	Linear functional.

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Abstract

This thesis deals with the study of singularity perturbed ordinary differential equation, and which considered a basis for obtaining the system of differential algebraic equations. In this study the extended system reduction method is used to reduce the ODEs to an enlarged system(extended system). In the reduced system we have seen the bifurcation point of the singularly perturbed system will become nonsingular equilibrium point and the Jacobian matrix of enlarged system is nonsingular,((we can use any numerical method to compute this nonsingular equilibrium point which in turn is a bifurcation point of the original system)). We discussed the way to deal with perturbed parameter for singularly perturbed ODEs through the study of the two cases:

Case 1:Include the case of perturbed parameter when $0 < \epsilon \ll 1$ i.e we have study the investigation of bifurcation point of singularly ODEs:

$$\begin{aligned}\dot{x} &= f(x, z, \epsilon) \\ \epsilon \dot{z} &= g(x, z, \epsilon)\end{aligned}$$

and this system will be reduced to an enlarged system by using extended system reduction.

Case 2:Include the case when perturbed parameter ϵ equal to zero. Then system of differential algebraic equation will be obtained:

$$\begin{aligned}\dot{x} &= f(x, z, \mu, 0) \\ 0 &= g(x, z, \mu, 0),\end{aligned}$$

where μ is bifurcation parameter.

This system will be reduced to an enlarged system by extended system reduction method and also the bifurcation theory will be applied to the system.

Keywords: Singularity Perturbed ODEs, Differential-Algebraic Equations, Extended System Reduction Method, Bifurcation Theory.

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Chapter 1

Introduction

1.1 Interview

This thesis consist of:

- Study of singularly perturbation ODEs.
- Study differential-algebraic equations of singularly perturbed ODEs .

Singularly perturbed ordinary differential equations of the form:

$$\begin{aligned}\dot{x} &= f(x, z, \mu) \\ \epsilon \dot{z} &= g(x, z, \mu), \quad x \in R^n, z \in R^m, \mu \in R.\end{aligned}$$

Date of this system seems to find the approximate solution through the application of perturbation theory at the parameter small. established the foundations of the Perturbation theory by Poincar and Stieltjes and then studied in ninetieth century, has worked a lot in the application of this field, the purpose of perturbation problem is to use the limiting behavior of the system when perturbation parameter approaches to zero[1].

M.Holmes[2] in his book studied dealing with many of the perturbation methods currently used by mathematician, scientists, and engineers.

Mathematication studied the branch that deals with singular limits which

is called singular perturbation theory that is one of the important types of perturbation theory, where: In 1979 the study of the asymptotic analysis of singular perturbation which works on simple ordinary equation that can be solved exactly [3].

Xiao-Biao Lin [4], in 1989, studied a heteroclinic bifurcation of singularly perturbed boundary value problems by presenting a complete procedure to compute higher order expansions with studying the existence of periodic solutions (relaxation oscillations) in order to compute the heteroclinic bifurcation.

Leonid S. Frank [5], in 1990, studied singular perturbations on manifolds without boundary by using the functional analysis.

E.M. de Jager and J.F. Furu [6] studied the singular perturbation problems of linear ordinary differential equation.

Verhulst and Ferdinand [7] studied applications to the natural and engineering sciences which is the main focus of this workbook, in a systematic introductory manner where delineates boundary layer theory for ordinary and partial differential equations, multi-timescale phenomena for nonlinear oscillations, diffusion and nonlinear wave equations.

Jeremy Warren R. [8], in 2012, stated the main three Fenichel's Theorems and its applications in dynamical systems of the geometric singular perturbation theory.

The singularly perturbed ODEs achieve when to be $0 < \epsilon \ll 1$ but when ϵ approach to zero resulting a system is often a DAEs. This thesis is devoted to investigate the sufficient conditions for the bifurcation to be occurred in singularity perturbed ODEs when perturbed parameter $0 < \epsilon \ll 1$, then this implies the study of the bifurcation of solution for differential algebraic equations.

Differential algebraic equations appear in the composition of applications, so their analysis and numerical treatment play an important role in modern mathematics, developing mathematical programs robustness in the application

of numerical method, analysis and formulation and solving systems of differential algebraic equations establish problems in science and engineering. In the past decades, DAEs have been an important subject in many system modelling and analysis in different spaces such as:

Constrained mechanics, chemical processes, Computational Fluid Dynamics (CFD), including nonlinear electric and gas transport networks, in addition circuit theory, electronic circuit theory and control theory [9].

F. R. Gantmacher [10] stated the analysis of the linear systems of differential equations in terms of matrices and their pencils with possibly singular leading coefficient matrix. The differential algebraic equation was applied in many fields including electronic circuit theory in work Ricardo Riasa [11]. Also DAEs applied the Numerical Methods in Multibody Dynamics at work (soellner,Claus[12]).

The power in application that is responsible for DAEs nowadays is considered as an established field in applied mathematics and subject of various monographs and textbooks as in(Kunkel,V. Mehrmann[9],Brenan,Campbell, Petzold[13]; Campbell[14]). Many researchers have worked on differential algebraic equations and ordinary differential equation presented in applied mathematics and computation,where: Michael[15] int his contributes to the qualitative theory of differential-algebraic equations (DAEs) by providing new stability criteria for solutions of a class of nonlinear, fully implicit DAEs with a properly stated derivative term and tractability index one and two. In [16] studied efficient method to compute singular points and singularity induced bifurcation points of differential-algebraic equations (DAEs)for a multimachine power system model.

Ibrahim [17] studies bifurcation theory about transcritical and pitchfork in DAEs without reduction.

Zahraa [18] discusses bifurcation in differential algebraic equation system obtain from singularity perturbed ODEs by reducing them by center manifold

method when the parameter $0 < \epsilon \ll 1$.

Many mathematical models for power systems used differential algebraic equations (DAEs), from general form as:

$$F(\dot{x}, x, \mu) = 0 \quad (1.1.1)$$

this equation can be also modelling by other the models that depend on the phenomena one wishes to investigate. [19].

1.2 Previous Works on Extended System Reduction Method

Keller [20] presented two key ideas which have provided the basis for much of the subsequent of numerical computation of bifurcation problems, the first idea is that of continuation and path following and the second idea showed how to extend the system of equations under investigation by adding an additional parameter. The ideas of Keller are subsequently developed and expanded by a number of authors in the late 1970s and 1980s. Extended systems were proposed to compute limit points in work Moore and Spence [21], A. Griewank and G. Reddien [22] for bifurcation points and constrained optima are presented. All the proposed extended systems have the property that they are regular at their target bifurcation points. At first these methods were applied to small systems of algebraic equations, and to equations arising from the discretisation of ordinary differential equations. However, it became apparent that they could be applied equally well to discretisations of partial differential equations. In the early 1980s a number of papers appeared in which the ideas of continuation and path following, together with the use of extended systems, were applied to problems in fluid mechanics.

Keener and Keller [23] used the extended system reduction to determine turning point arising from perturbed bifurcation.

G.ponisch[24] this paper deals with the computation of simple bifurcation point of non-linear parameter dependent equation by using a minimally extended system.

Hu Guo-qing [25]discussed two extended system to deal with non simple turning points for two parameter problems.

K. Kane; B. J. Torby [26] studied the existing Modal Reduction Method, which was developed to handle symmetric mass and stiffness matrices,

Kamal[27] discussed the stability of an equilibrium solution of ODE effected by extended system.

The extended system method provide may be used for equation systems in one, two or three dimensions. However, in order to present the basic idea did not be only look at the numerical issue of the equation in one dimension[28].

Kamal [19]where the extended system reduction method used to compute singularity induced bifurcation points with one parameter.

1.3 Our Work

Differential algebraic equations (DAEs) are studied through singularity perturbed ODEs. In this study the first case, when perturbation parameter $0 < \epsilon \ll 1$ that is the singularity perturbed ODEs by using extended system reduction method resulting in an extended system of equations.

This will permit us to apply the classical bifurcation theory on singularity perturbed ODEs for the new equation. So, this will be shown by giving sufficient conditions for saddle-node, pitchfork and transcritical bifurcation also happens in singularly perturbed equation. Including the second case when perturbed parameter ϵ equal to zero. Then system of differential algebraic equation will be obtained, This system will be reduced to an enlarged system by extended system reduction method and also the bifurcation theory will be

applied to differential algebraic equation.

1.4 Construction of this thesis

The construction of the thesis is as follows:

chapter 1 includes introduction and previous studies some important concepts and definitions of this thesis.

Chapter 2 in this chapter we will work to studying extended system reduction to obtain reduced model from singularity perturbed ODEs when perturbation parameter $0 < \epsilon \ll 1$, before that introducing general idea about singularity perturbed ODEs and extended system reduction. Later we will introduce theories of bifurcation on singularity perturbed ODEs by using extended system reduction when parameter $0 < \epsilon \ll 1$.

Chapter 3: This chapter is devoted to study differential algebraic equations by using the extended system reduction method when perturbed ϵ equal to zero, and then applying the bifurcation theory on the resulting DAEs in one dimensional when μ is bifurcation parameter.

Chapter 4: Conclusion of this thesis and suggested future work related to this topic.

1.5 Definitions and concepts

In this section some definition and concepts related to our work will be introduced.

1.5.1 Singularly perturbed ODEs [29]

Singular perturbation is a differential equation with another conditions having a small parameter that is multiplying the highest derivatives. The fundamental reason for studying singularity perturbed theory is to consider a problem with a small parameter ϵ and state a solution $x(\mu, \epsilon)$. A singular perturbation occurs whenever the limit of regular perturbation problem fails. Standard form of singularly perturbed ODEs:

$$\begin{aligned}\dot{x} &= f(x, z, \mu, \epsilon) \\ \epsilon \dot{z} &= g(x, z, \mu, \epsilon)\end{aligned}$$

where $f : R^n \times R^m \times R \times R \rightarrow R^n, g : R^n \times R^m \times R \times R \rightarrow R^m$. Then the differential-algebraic equation will be obtained by setting $\epsilon = 0$

$$\begin{aligned}\dot{x} &= f(x, z, \mu, 0) \\ 0 &= g(x, z, \mu, 0),\end{aligned}$$

where $f : R^n \times R^m \times R \rightarrow R^n, g : R^n \times R^m \times R \rightarrow R^m$.

1.5.2 Some Types of Differential Algebraic Equation(DAEs)

1. Non-Linear DAEs [30, 31]: Non-linear DAEs occur for example, which the modelling electrical circuits or coupled mechanical systems is used throughout analysis.

This type consists of:

- Fully Implicit: The form for this type is:

$$F(\dot{x}, x, \epsilon) = 0$$

- Semi-explicit: The form of this type is:

$$\begin{aligned}\dot{x} &= f(x, z, \epsilon) \\ 0 &= g(x, z, \epsilon)\end{aligned}$$

where x is differential variable, z is algebraic variable.

2. Hessenberg DAEs [32] : In this type the form as written :

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \dots, x_k, t) \\ \dot{x}_2 &= f_2(x_1, \dots, x_{k-1}, t) \\ \dot{x}_3 &= f_3(x_1, \dots, x_{k-2}, t) \\ &\vdots \\ \dot{x}_i &= f_i(x_1, \dots, x_{k-i}, t), \quad 2 \leq i \leq k-1 \\ &\vdots \\ 0 &= f_k(x_1, \dots, x_k)\end{aligned}$$

where the matrix $k \times k$, $\frac{\partial f_k}{\partial x_{k-1}} \frac{\partial f_{k-1}}{\partial x_{k-2}} \dots \frac{\partial f_2}{\partial x_1} \frac{\partial f_1}{\partial x_k}$ non singular.

3. Linear time-invariant DAEs[11]: Linear type used in problem-solving DAE which writes as :

$$A(t)\dot{x} + B(z, t) = q(t)$$

where $A(t)$ and $B(t)$ are $n \times n$ matrices.

4. Quasilinear DAEs[11]: This type has the form

$$A(t)\dot{x} = h(x)$$

where $A : W_0 \rightarrow R^{m \times m}$, $h : W_0 \rightarrow R^m$ are sufficiently smooth mappings and W_0 is open set in R^m .

1.5.3 Index concept

The concept of the index for the first time appeared in determining time-varying of linear systems and regulations implied public in recent years. The smoothness requirements can be determined from the Kronecker canonical form of the matrix pair. It has been used to measure the distance from differential algebraic equations with ordinary differential equations, which provided information on the structure of the mathematical and complications in the analytical solution of differential algebraic equation[13]. Consider the special form of a semi explicit DAEs :

$$\begin{aligned}\dot{x} &= f(x, z, \mu) \\ 0 &= g(x, z, \mu)\end{aligned}$$

with the derivation of constraint equation:

$$\frac{\partial g}{\partial x}(x, z, \mu)\dot{x} + \frac{\partial g}{\partial z}(x, z, \mu)\dot{z} = -\frac{\partial g}{\partial \mu}(x, z, \mu)\dot{\mu}$$

If $(\frac{\partial g}{\partial z})^{-1}$ is nonsingular the DAEs is of index one otherwise we continue differentiation of the constraint, we get DAEs with index 2 and upper. Differential algebraic equations owns index one while ordinary differential equations possess index zero. In addition, there are many types of index concept mention some of them:

- Differentiation index [33]: The general form

$$F(\dot{x}, x, t) = 0$$

is the minimum number of differentiation

$$\frac{\partial}{\partial t}F(\dot{x}, x, t) = 0, \dots, \frac{\partial^l}{\partial t^l}F(\dot{x}, x, t) = 0$$

where l is the differential index of nonlinear sufficiently smooth DAEs.

- Perturbation index[34] : The perturbation index introduced for nonlinear DAEs

$$F(\dot{x}(t), x(t)) = 0.$$

It introduced the concept of perturbation index of considerable importance in the numerical treatment of differential algebraic equations that have been used to measure the sensitivity of solutions with regard to concern the problem.

- Strangeness index [33]: It is a concept closely linked to the concept of differentiation index and extends to excessive. The following example applies to the differentiation index:

Example 1.5.1. [35] Consider the DAEs

$$\begin{aligned}\dot{x} &= f(x, z) \\ 0 &= g(x, z)\end{aligned}$$

we differential the constraint w.r.t t

$$\begin{aligned}0 &= g_x \dot{x} + g_z \dot{z} \\ 0 &= g_x f + g_z \dot{z} \\ \dot{z} &= -g_z^{-1} g_x f \\ \dot{x} &= x + 1 \\ 0 &= (x + 1)z + 2 \\ \dot{z} &= -g_z^{-1} g_x f \\ &= -(x + 1)^{-1} z f \\ &= -\frac{z(x + 1)}{x + 1} = -z\end{aligned}$$

Then DAEs is of index one.

1.5.4 Implicit Function Theorem [36]

This theorem is very important in this thesis, which is the first step in the work of singularity perturbed ordinary differential equations in Chapter two. Consider the equation:

$$f_i(x_1, \dots, x_m, \mu_1, \dots, \mu_m) = 0, \quad i = 1, 2, \dots, m$$

And written as in vector form :

$$F(x, \mu) = 0, \tag{1.5.1}$$

$$F(x, \mu) = \begin{pmatrix} f_1(x_1, \mu_1) \\ f_2(x_2, \mu_2) \\ \vdots \\ f_m(x_m, \mu_m) \end{pmatrix}$$

where $F : R^m \times R^m \rightarrow R^m$ and the Jacobian matrix of this form :

$$G_y(x, \mu) = \left(\frac{\partial f_i}{\partial x_j}(x, \mu) \right)_{i,j=1,\dots,m}$$

which must be nonsingular.

Theorem 1.5.1. [36] (*Implicit Function theorem*)

Consider F the function, suppose that

$$F(x_0, \mu_0) = 0$$

and that

$$\det G_y(x_0, \mu_0) \neq 0.$$

Then there exists neighbourhood W_1 of x_0 in R^m and W_2 of μ_0 in R^m and a function $X : W_1 \rightarrow W_2$ such that for every $\mu_0 \in W_2$, that has a unique solution $x = X(\mu)$ in W_1 , if $F \in C(R^m)$, then $X \in C(R^m)$. i.e. $F(X(\mu), \mu) = 0$

1.5.5 Equilibrium Point

In dynamic systems solutions can be found to problems of linear systems directly while in general, life problems only revolve on nonlinear systems..

In mathematics, specifically in differential equations, an equilibrium point is a constant solution to a differential equation. Equilibria can be classified by looking at the signs of the eigenvalues of the linearization of the equations about the equilibria. That is to say, by evaluating the Jacobian matrix at each of the equilibrium points of the system, and then finding the resulting eigenvalues, the equilibria can be categorized. Then the behavior of the system in the neighborhood of each equilibrium point can be qualitatively determined, (or even quantitatively determined, in some instances), by finding the eigenvector associated with each eigenvalue[37].

If all eigenvalues have negative real part, the equilibrium is a stable node. If at least one has a positive real part, the equilibrium is an unstable node. If at least one eigenvalue has negative real part and at least one has positive real part, the equilibrium is a saddle point[38].

The equilibrium is said to be hyperbolic if all eigenvalues of the Jacobian matrix have non zero real parts. While If at least one eigenvalues of the Jacobian matrix is zero or has a zero real part, then the equilibrium is said to be non-hyperbolic[39].

1.5.6 Bifurcation Theory [37]

Bifurcation theory studies the change in behavior of the system with the change in parameters that involves the change in the dynamics behavior. These changes are only qualitative in nature. But there may be changes in situations as well. The bifurcation point is defined as the sudden change in the behavior of the system when a parameter passes through a critical value. In bifurcation point, at least one eigenvalue of the Jacobian matrix gets zero-

real part

In the problems of bifurcation, there are two types used in the studying the space of the state control:

- **Global Bifurcation**[39]

This type of bifurcation occurs when the largest constant groups, for example periodic orbits collide with the equilibria as this changes the trajectories in the phase space.

- **Local bifurcation**[37]

A local bifurcation occurs when a parameter change causes the stability of an equilibrium point when the Jacobian matrix has an eigenvalue with zero real part.

In this thesis, three basic kinds of local bifurcation Fig.(1.1) will be applied:

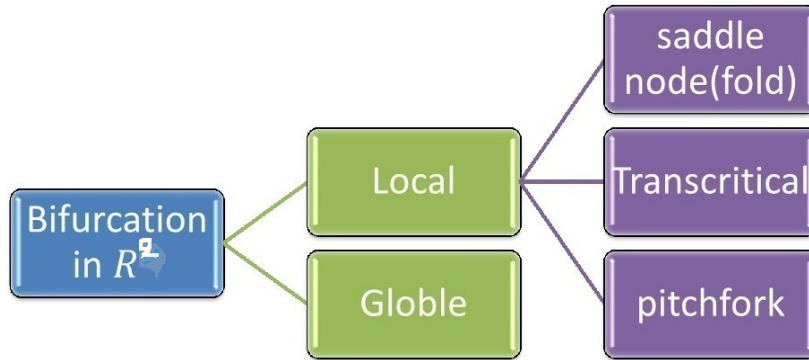


Figure 1.1: The kinds of bifurcation.

1. **Fold(saddle-node) bifurcation** [40]

In the saddle-node bifurcation a saddle equilibrium collide and disappear the parameter also changes, A unique curve (or branch) of critical points

passed through the origin that is tangent to line $\mu = 0$ at $x = 0$. Fig.(1.2) [37] illustrates the bifurcation in a simple model. Usually they are in the form:

$$\dot{x} = \mu - x^2$$

$$\dot{z} = -z$$

, $(x, z) \in R^2, \mu \in R$. The sufficient conditions to the occurrence of a fold bifurcation point are calculated by:

$$\mu = -x^2$$

$$\frac{\partial \mu}{\partial x} = -2x = 0$$

$$\frac{\partial^2 \mu}{\partial x^2} = -2 \neq 0$$

2. Transcritical bifurcation[37]

Transcritical bifurcation is one in which an equilibrium point exists for all values of a parameter and is never destroyed, where it has one point that is unstable and all other points are stable. Two curves of critical points intersect at the origin in the $\mu - x$ plane. Both curves existed on either side of $\mu = 0$. Fig.(1.3) [37] illustrates the points of this bifurcation. It is described usually for one variable by the form:

$$\dot{x} = \mu x - x^2$$

$$\dot{z} = -z$$

, $(x, z) \in R^2, \mu \in R$. The sufficient conditions to the occurrence of a Transcritical bifurcation point are calculated by:

$$\mu x = -x^2$$

$$\mu = x$$

$$\frac{\partial \mu}{\partial x} = 1 \neq 0$$

3. Pitchfork bifurcation[37]

The Pitchfork bifurcation is known as the transition from one stable equilibrium to a new stable equilibrium, two separated by a saddle bifurcation. Two curves of fixed points intersected at the origin in the $\mu - x$ plane. Only one curve ($x = 0$) existed on both sides of $\mu = 0$ is described usually for one variable in Fig.(1.4) [37] illustrates this bifurcation by:

$$\dot{x} = \mu x - x^3$$

$$\dot{z} = -z$$

$$(x, z) \in R^2, \mu \in R$$

the sufficient conditions to the occurrence Pitchfork bifurcation point calculated by:

$$\mu x = -x^3$$

$$\mu = x^2$$

$$\frac{\partial \mu}{\partial x} = -2x = 0$$

$$\frac{\partial^2 \mu}{\partial x^2} = -2 \neq 0$$

We will apply the bifurcation theory on extended system reduction of small parameter in following chapters.

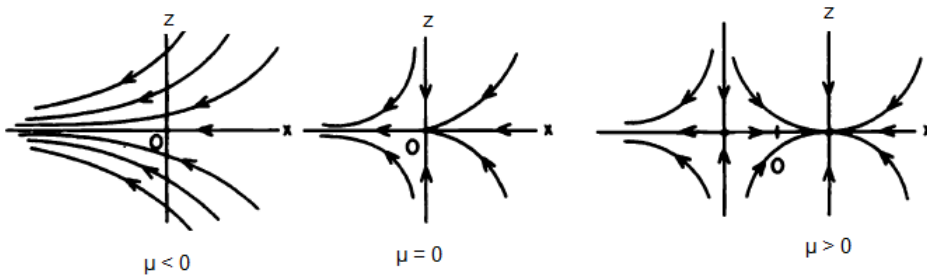


Figure 1.2: The fold bifurcation for $\mu < 0, \mu = 0, \mu > 0$.

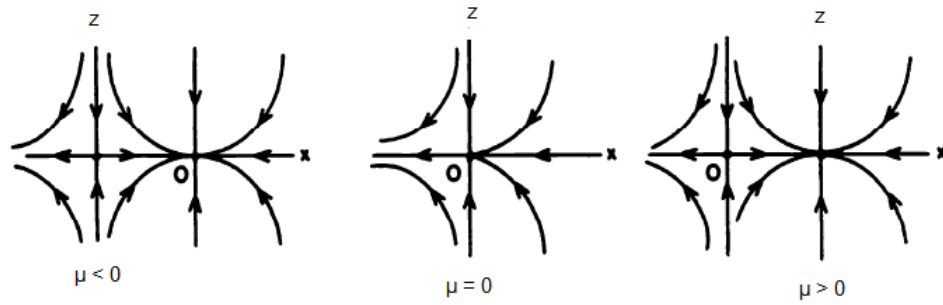


Figure 1.3: The transcritical bifurcation for $\mu < 0, \mu = 0, \mu > 0$.

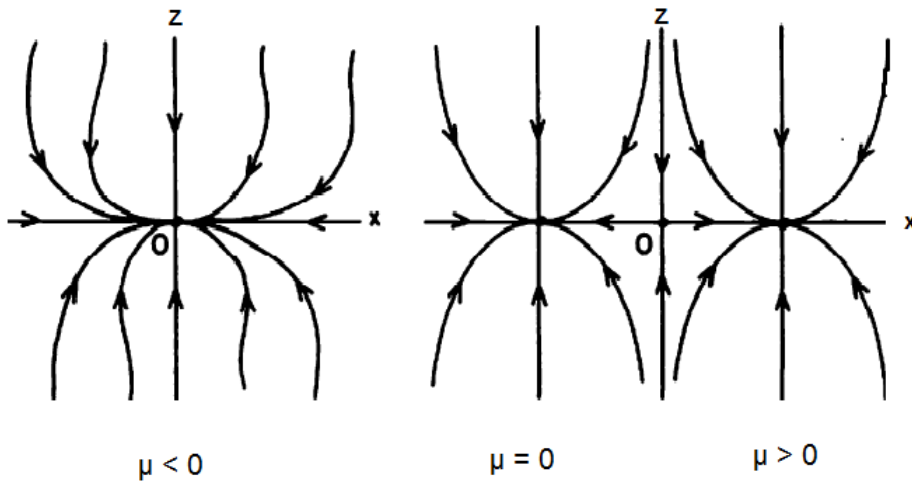


Figure 1.4: The pitchfork bifurcation for $\mu < 0, \mu = 0, \mu > 0$.

Chapter 2

Bifurcation of Solution in Singularity Perturbed ODE by Extended System Reduction Method

2.1 Introduction

In this chapter, singularly perturbed ODEs are studying by using extended system reduction reduced to an enlarged system when the perturbed parameter $0 < \epsilon \ll 1$. This will permit us to apply the classical bifurcation theory on singularly parameterized ODEs. So, this will be shown by giving sufficient conditions for saddle-node, transcritical and pitchfork bifurcation to be occurred in such system. We introduce concept on singularly perturbed ODEs and explain the reduction method to obtain the reduced model from the singularly perturbed ODEs. We discuss extended system reduction of singularly perturbed ODEs when the perturbed parameter $0 < \epsilon \ll 1$, discuss bifurcation

theory on system obtained by the extended system reduction when ϵ bifurcation parameter.

2.2 Extended System Reduction of singularly perturbed ODEs

Singularly perturbed systems is a special class of nonlinear control systems. singular problem of a dynamical system refers to the derivatives some of the states that are multiplied by a small positive parameter ϵ [41].

The aim of studying singularly perturbation problem is to use the limiting behavior of the system when perturbation parameter ϵ approaches to zero, in order to get an idea of what the system looks like when ϵ is small[18].

This chapter is devoted to investigate the sufficient conditions for the bifurcation to be occurred in singularly perturbed ODEs when perturbed parameter $0 < \epsilon \ll 1$.

We will consider the singularly perturbed ODEs has the standard form:

$$\begin{aligned} \dot{x} &= f(x, z, \epsilon) \\ \epsilon \dot{z} &= g(x, z, \epsilon), \quad x \in R, z \in R, \epsilon \in R, \end{aligned} \quad (2.2.1)$$

and $0 < \epsilon \ll 1$ is singularly perturbed parameter and will be a bifurcation parameter.

Consider the points $(x, z, \epsilon) \in R \times R \times R$ such that :

$$F(x, z, \epsilon) = \begin{pmatrix} f(x, z, \epsilon) \\ \frac{1}{\epsilon}g(x, z, \epsilon) \end{pmatrix} = 0, \quad (2.2.2)$$

where the vector $(f, g) : R \times R \times R \longrightarrow R \times R \times R$, $0 < \epsilon \ll 1$ and $\epsilon \in R$, with define the singular surface:

$$\mathcal{S} = \{(x_*, z_*, \epsilon_*) \in R \times R \times R : f(x_*, z_*, \epsilon_*) = g(x_*, z_*, \epsilon_*) = 0, \text{rank} DF(x_*, z_*, \epsilon_*) = 0\}$$

then by using extended system reduction, we may associate such perturbed equilibrium solution to obtain the equation:

$$G(x, z, \epsilon, \phi) = 0 \quad (2.2.3)$$

where G define a computationally convenient system which characterizes fold curves and which provides existence results[28]. The following is definition of extended system reduction.

Definition 2.2.1. [28] Consider the system

$$F(x, z, \epsilon) = 0 \quad (2.2.4)$$

The extended system reduction of the singularly perturbed ODEs (2.2.1) results in the reduced system

$$G(x, z, \epsilon, \phi) = \begin{pmatrix} F(x, z, \epsilon) \\ F_x(x, z, \epsilon)\phi \\ F_z(x, z, \epsilon)\phi \\ L^\top \phi - 1 \end{pmatrix} = 0 \quad (2.2.5)$$

where $\phi \in R$ and L linear functional will be chosen for some nonzero constant vector to ϕ , $L^\top \phi = 1$. The Jacobian matrix of G :

$$DG_y \begin{pmatrix} F_x & F_z & F_\epsilon & 0 \\ F_{xx}\phi & F_{xz}\phi & F_{x\epsilon}\phi & F_x \\ F_{zx}\phi & F_{zz}\phi & F_{z\epsilon}\phi & F_z \\ 0 & 0 & 0 & L^\top \end{pmatrix} \quad (2.2.6)$$

where $y = (x, z, \epsilon, \phi)$, thus we can prove the Jacobian matrix of the reduced system at equilibrium point $(x_0, z_0, \epsilon_0, \phi_0)$ is nonsingular as in Theorem 2.2.2.

The following results will be discussed in this chapter:

- The bifurcation points of the system (2.2.1) will be associated with the equilibrium points of the extended system reduction(2.2.5).

- We will show the bifurcation points of the system (2.2.1) are associated with the non degenerated equilibrium points of extended system reduction (2.2.5), if certain non degeneracy conditions on functions f and g are satisfied.
- conditions on f and g will be given such that the basic type of bifurcation saddle-node, transcritical and pitchfork bifurcation will be discussed.
- An iterative method (e.g Newton-raphson) we can be applied to compute the zeroes of the extended system reduction (2.2.5) which in turn include bifurcation points of system (2.2.1).

Theorem 2.2.1. The point (x_0, z_0, ϵ_0) is a bifurcation point of system (2.2.1) iff the point $(x_0, z_0, \epsilon_0, \phi_0)$ is an equilibrium point of extended system reduction (2.2.5).

Proof. Suppose that (x_0, z_0, ϵ_0) is bifurcation point of system (2.2.1) then from rank condition we conclude that $DF(x_0, z_0, \epsilon_0)\phi_0 = 0$ for some eigenvector ϕ_0 and $L^\top \phi_0 - 1 = 0$ for some chosen vector $L^\top$. Hence $(x_0, z_0, \epsilon_0, \phi_0)$ is equilibrium point of extended system reduction on other hand if $(x_0, z_0, \epsilon_0, \phi_0)$ is equilibrium point of system (2.2.5) then $F(x_0, z_0, \epsilon_0) = 0$ and rank condition is satisfied which mean that (x_0, z_0, ϵ_0) is bifurcation point. $\square$

Theorem 2.2.2. The Jacobian matrix G_y^0 is nonsingular if

1. $F_\epsilon^0(x_0, z_0, \epsilon_0) \notin \text{range}(F_x, F_z)$.
2. $\text{range } F_x^0(x_0, z_0, \epsilon_0) \cap \text{range } F_z^0(x_0, z_0, \epsilon_0) = \{0\}$ and $\ker F_z^0(x_0, z_0, \epsilon_0) = \text{span}\{\phi\}$, $\ker F_x^0(x_0, z_0, \epsilon_0) = \text{span}\{\psi\}$, $0 \neq \psi \in R$.
3. $F_z^0\phi(x_0, z_0, \epsilon_0) \notin \text{range } F_x^0(x_0, z_0, \epsilon_0)$
4. $F_{zx}^0(x_0, z_0, \epsilon_0)\phi\psi \notin \text{range } F_z^0(x_0, z_0, \epsilon_0)$.

Proof. We shall prove that $\ker G_y^0 = \{0\}$, where G is extended system reduction of F .

$$G_y^0(x, z, \epsilon, \phi) = \begin{pmatrix} F_x^0 & F_z^0 & F_\epsilon^0 & 0 \\ F_{xx}^0 \phi & F_{xz}^0 \phi & F_{x\epsilon}^0 \phi & F_x^0 \\ F_{zz}^0 \phi & F_{zx}^0 \phi & F_{z\epsilon}^0 \phi & F_z^0 \\ 0 & 0 & 0 & L^\top \end{pmatrix}$$

such that “0” denotes evaluation at $(x_0, z_0, \epsilon, \phi_0)$. Let $(u, v, \mu, \omega) \in \ker G_y^0$.

Then

$$G_y^0 \begin{bmatrix} u \\ v \\ \mu \\ \omega \end{bmatrix} = \begin{pmatrix} F_x^0 u + F_z^0 v + F_\epsilon^0 \mu \\ F_{xx}^0 u \phi + F_{xz}^0 v \phi + F_{x\epsilon}^0 \mu \phi + F_x^0 \omega \\ F_{zx}^0 \phi u + F_{zz}^0 \phi v + F_{z\epsilon}^0 \mu \phi + F_z^0 \omega \\ L^\top \omega \end{pmatrix} = 0$$

this implies

$$F_x^0 u + F_z^0 v + F_\epsilon^0 \mu = 0$$

From condition (1) we get $\mu = 0$ this given:

$$F_x^0 u + F_z^0 v = 0.$$

$$F_{xx}^0 u \phi + F_{xz}^0 v \phi + F_x^0 \omega = 0.$$

$$F_{zx}^0 \phi u + F_{zz}^0 \phi v + F_z^0 \omega = 0.$$

$$L^\top \omega = 0.$$

According to the condition(2) we get $F_z^0 v = 0$ and $v = \alpha \phi$ $\alpha \in \mathbb{R}$, $\phi \neq 0$, so substituting into equations above:

$$F_x^0 u + \alpha F_z^0 \phi = 0.$$

$$F_{xx}^0 u \phi + \alpha F_{xz}^0 \phi \phi + F_x^0 \omega = 0.$$

$$F_{zx}^0 \phi u + \alpha F_{zz}^0 \phi \phi + F_z^0 \omega = 0.$$

$$L^\top \omega = 0.$$

From $F_x^0 u + \alpha F_z^0 \phi = 0$ and condition(3) we have $\alpha = 0$, so $F_x^0 = 0$ from condition:

$$\ker F_x^0(x_0, z_0, \epsilon_0) = \text{span}\{\psi\}$$

this give $u = \beta\psi$, $\beta \in \mathbb{R}$, $\psi_0 \neq 0$ substituting into:

$$\beta F_x^0 \psi = 0.$$

$$\beta F_{xx}^0 \phi \psi + F_x^0 \omega = 0.$$

$$\beta F_{zx}^0 \phi \psi + F_z^0 \omega = 0.$$

$$L^\top \omega = 0$$

From the equation $\beta F_x^0 \psi = 0$, $\beta = 0$ so $u=0$

$$F_x^0 \omega = 0, F_z^0 \omega = 0$$

$$L^\top \omega = 0$$

by condition

$$\ker F_z^0(x_0, z_0, \epsilon_0) = \text{span}\{\phi\}$$

so $\omega = \gamma\phi$ this given

$$\gamma F_x^0 \phi = 0$$

$$\gamma F_z^0 \phi = 0$$

$$\gamma L^\top \phi = 0.$$

this imply $\gamma = 0$ hence $\omega = 0$

□

Example 2.2.1. Consider the singularly perturbed ODEs:

$$\begin{aligned}\dot{x} &= xz \\ \epsilon \dot{z} &= z - \epsilon x\end{aligned} \quad (x, z) \in R^2, \epsilon \in R.$$

It easy show $P = (0, 0, \epsilon)$ is singular equilibrium point in which $\frac{\partial g}{\partial z} = 1 \neq 0$ and as well we have $\phi_0 = (\phi_1, \phi_2) = (1, 0)^\top$ and L chosen to be $L^\top = (L_1, L_2) = (1, 0)^\top$ then by extended system reduction the system above is:

$$G(0, 0, \epsilon, \phi) = \begin{bmatrix} xz \\ z - \epsilon x \\ z\phi_1 \\ -\epsilon\phi_2 \\ x\phi_1 \\ \phi_2 \\ (L_1\phi_1 + L_2\phi_2) - 1 \end{bmatrix}$$

It easy show $P = (0, 0, \epsilon, \phi)$ is equilibrium point of extended system reduction. Then according to $F(x, z, \epsilon) = 0$ and $F_x(x, z, \epsilon)\phi = 0$, we conclude that $(0, 0, \epsilon)$ is bifurcation point. By the conditions of Theorem 2.2.2 above for the point P we have got:

$$\text{range} \begin{pmatrix} F_x & F_z \end{pmatrix} (0, 0, \epsilon) = \left\{ \alpha \begin{pmatrix} 0 \\ -\epsilon u + v \end{pmatrix} : \alpha \in R \right\},$$

and

$$F_\epsilon(0, 0, \epsilon) = \begin{pmatrix} 0 \\ -x \end{pmatrix} = 0$$

obviously the condition (1) is satisfy.

$$F_z(0, 0, \epsilon)\phi_0 = \begin{pmatrix} -x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\text{range } F_x(0, 0, \epsilon) = \{ \beta(zu - \epsilon v) = -\epsilon v : \beta \in R \}.$$

Therefore the condition (3) is satisfy.

The condition (4) also satisfy where

$$F_{zx}(0, 0, \epsilon)\phi_0\psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_0 = (1, 0)^\top$$

and

$$\text{range } F_z(0, 0, \epsilon) = \{\gamma(xu + v) = v : \gamma \in R\}.$$

Hence the point $P(0, 0, \epsilon)$ is associating with the non degenerate equilibrium point $(0, 0, \epsilon, \phi)$ of extended system reduction.

Then the Jacobian matrix as:

$$DG(0, 0, \epsilon, \phi) = \begin{pmatrix} z & x & 0 & 0 \\ -\epsilon & 1 & -x & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & -1 & -\epsilon \\ 1 & 0 & 0 & x \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & L^\top \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -\epsilon \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is nonsingular of equilibrium point of extended system reduction which ensures the conclusion of Theorem 2.2.2.

2.3 Bifurcation of solution by extended system reduction method

In this section, we will use extended system reduction to solve bifurcation problem for which we can compute bifurcation points, which is correspond to find the associated nonsingular point of the extended system reduction.

The sufficient conditions to prove the kinds bifurcation (fold, transcritical, pitchfork) in following theorems at ϵ is bifurcation point. We consider the

following system of equations:

$$F(x, z, \epsilon) = \begin{bmatrix} f(x, z, \epsilon) \\ \frac{1}{\epsilon}g(x, z, \epsilon) \end{bmatrix} = 0 \quad (2.3.1)$$

where $F : R \times R \times R \longrightarrow R$, and ϵ a bifurcation parameter.

We will assume (x_*, z_*, ϵ_*) is an equilibrium point so we have:

$$F(x_*, z_*, \epsilon_*) = \begin{bmatrix} f(x_*, z_*, \epsilon_*) = 0 \\ \frac{1}{\epsilon}g(x_*, z_*, \epsilon_*) = 0, \end{bmatrix} \quad (2.3.2)$$

and the $rank DF(x, z, \epsilon) = 0$, by the implicit function theorem there exist a smooth branch to (2.2.5) provided the Jacobian matrix of the reduced system (2.2.6) is nonsingular as we proved in Theorem 2.2.2.

The Jacobian matrix of the extended system reduction is non singular at equilibrium point when $0 < \epsilon \ll 1$ is given by:

$$G_y^* = \begin{pmatrix} F_x^* & F_z^* & F_\epsilon^* & 0 \\ F_{xx}^*\phi & F_{xz}^*\phi & F_{x\epsilon}^*\phi & F_x^* \\ F_{zx}^*\phi & F_{zz}^*\phi & F_{z\epsilon}^*\phi & F_z^* \\ 0 & 0 & 0 & L^\top \end{pmatrix} \quad (2.3.3)$$

“*” “is superscript indicate evaluated at $y = (x_*, z_*, \epsilon_*, \phi_*)$.”

2.3.1 Saddle-node(fold)bifurcation of singularly perturbed ODEs

A fold bifurcation point is a pair of equilibria, meets and disappears with a zero eigenvalue. One of the equilibria (saddle) is unstable while the other (node) is stable[37].

Theorem 2.3.1. Consider the system (2.3.2) and the extended system reduction (2.2.5) satisfy conditions of Theorem 2.2.2, if the following conditions hold:

1. $F(0, 0, \epsilon) = 0$.
2. $F_x(0, 0, \epsilon) = 0$.
3. $F_\epsilon(0, 0, \epsilon) \neq 0$.
4. $F_{xx}(0, 0, \epsilon) \neq 0$.

Then $(0, 0, \epsilon)$ is saddle-node bifurcation point of the system (2.2.1) iff $(0, 0, \epsilon, \phi)$ is nonsingular equilibrium point of extended system reduction (2.2.5).

Proof. : If $(x_*, z_*, \epsilon_*) = (0, 0, \epsilon)$ is saddle-node bifurcation point of system (2.2.1) then by Theorem 2.2.1 that $(x_*, z_*, \epsilon_*, \phi_*) = (0, 0, \epsilon, \phi)$ is equilibrium point of extended system reduction (2.2.5). From Theorem 2.2.2 we get $\ker G_y^*(0, 0, \epsilon, \phi) = 0$ this implies that $(x_*, z_*, \epsilon_*, \phi_*)$ is nonsingular equilibrium point of the extended system reduction.

Suppose $(0, 0, \epsilon, \phi)$ nonsingular equilibrium point of extended system reduction (2.2.5). Then from the equation (2.2.5) we have $F(x_*, z_*, \epsilon_*) = 0$ and $F_x(x_*, z_*, \epsilon_*)\phi = 0$, from this we conclude that $(0, 0, \epsilon)$ is bifurcation point. Now we will prove that $(0, 0, \epsilon)$ is fold bifurcation point, under the conditions

given in the theorem above. We want to prove that

$$\frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = 0,$$

and

$$\frac{\partial^2 \epsilon}{\partial x^2}(0, 0, \epsilon) \neq 0.$$

By implicit function theorem we have:

$$F(x, z, \epsilon(x, z)) = 0 \quad (2.3.4)$$

Differentiate the equation(2.3.4)w.r.tx at(0, 0, ϵ)and by substitute $\frac{\partial z}{\partial x} = 0$ at(0, 0):

$$\frac{\partial F}{\partial x}(0, 0, \epsilon) + \frac{\partial F}{\partial \epsilon} \frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = 0$$

and according the condition(2)with condition(1) we get:

$$\frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = -\left(\frac{\partial F}{\partial \epsilon}(0, 0, \epsilon)\right)^{-1}\left(\frac{\partial F}{\partial x}(0, 0, \epsilon)\right).$$

Then

$$\frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = 0$$

Now differentiating equation $F(x, z, \epsilon(x, z)) = 0$ twice w.r.t x :

$$\frac{\partial^2 F}{\partial x^2}(0, 0, \epsilon) + \frac{\partial F}{\partial \epsilon} \frac{\partial^2 \epsilon}{\partial x^2}(0, 0, \epsilon) + \frac{\partial^2 F}{\partial \epsilon^2} \frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = 0.$$

We apply the property $\frac{\partial \epsilon}{\partial x} = 0$ and from conditions(2),(3) we get:

$$\frac{\partial^2 \epsilon}{\partial x^2} = -\left(\frac{\partial F}{\partial \epsilon}(0, 0, \epsilon)\right)^{-1}\left(\frac{\partial^2 F}{\partial x^2}(0, 0, \epsilon)\right) \neq 0.$$

Then the property $\frac{\partial^2 \epsilon}{\partial x^2} \neq 0$ is hold.

Then(0, 0, ϵ)is saddle-node bifurcation point of singularly perturbed ODEs(2.2.1).

□

2.3.2 Transcritical Bifurcation of singularity perturbed ODEs

In Transcritical bifurcation there is an exchange of stability between two equilibrium points, where there is one unstable and the other is stable equilibrium point[37].

Theorem 2.3.2. Consider the system (2.3.2) and the extended system reduction (2.2.5) satisfy conditions of Theorem 2.2.2, if the following conditions hold:

1. $F_x(0, 0, \epsilon) = 0$.
2. $F_\epsilon(0, 0, \epsilon) = 0$.
3. $F_{x\epsilon}(0, 0, \epsilon) \neq 0$.
4. $F_{xx}(0, 0, \epsilon) \neq 0$.

Then $(0, 0, \epsilon) = (x_*, z_*, \epsilon_*)$ is Transcritical bifurcation point of the system (2.2.1) iff $(0, 0, \epsilon, \phi)$ is nonsingular equilibrium point of extended system reduction (2.2.5).

Proof. Let $(x_*, z_*, \epsilon_*) = (0, 0, \epsilon)$ is transcritical bifurcation point of system (2.2.1) then by Theorem 2.2.1 that $(x_*, z_*, \epsilon_*, \phi_*) = (0, 0, \epsilon, \phi)$ is equilibrium point of extended system reduction (2.2.5). From Theorem 2.2.2 we have to $(0, 0, \epsilon, \phi)$ is nonsingular of equilibrium point of extended system reduction (2.2.5).

Now we will prove that $(0, 0, \epsilon)$ is transcritical bifurcation point, under theorem conditions at the above. Since $F_\epsilon(0, 0, \epsilon) = 0$ so implicit function theorem cannot be applied, then a different strategy will be applied. F The function will be written as follows:

$$F(x, z, \epsilon) = \begin{bmatrix} xM(x, z, \epsilon) \\ xN(x, z, \epsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.3.5)$$

where M and N are define as follows:

$$M(x, z, \epsilon) = \begin{bmatrix} \frac{f(x, z, \epsilon)}{x} & x \neq 0 \\ f_x(x, z, \epsilon) & x = 0 \end{bmatrix}$$

and

$$N(x, z, \epsilon) = \begin{bmatrix} \frac{1}{\epsilon} \frac{g(x, z, \epsilon)}{x} & x \neq 0 \\ \frac{1}{\epsilon} g_x(x, z, \epsilon) & x = 0 \end{bmatrix}$$

we need to prove that

$$\frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) \neq 0.$$

Now since $F_{x\epsilon}(0, 0, \epsilon) \neq 0$ so $M_\epsilon(0, 0, \epsilon) \neq 0$, then by implicit function theorem

$M(x, z, \epsilon(x, z)) = 0$ and differentiation w.r.t x at $(0, 0, \epsilon)$ we get:

$$\frac{\partial M}{\partial x}(0, 0, \epsilon) + \frac{\partial M}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial x}(0, 0) \right) = 0$$

Then

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(0, 0, \epsilon) + \frac{\partial^2 f}{\partial x \partial \epsilon}(0, 0, \epsilon) \frac{\partial \epsilon}{\partial x}(0, 0) &= 0, \\ \frac{\partial \epsilon}{\partial x}(0, 0) &= - \left(\frac{\partial^2 f}{\partial x \partial \epsilon}(0, 0, \epsilon) \right)^{-1} \left(\frac{\partial^2 f}{\partial x^2}(0, 0, \epsilon) \right), \end{aligned}$$

which from condition(3)and(4) we have:

$$\frac{\partial \epsilon}{\partial x}(0, 0) \neq 0.$$

Applying the same way on $N(x, z, \epsilon) = 0$ at $(0, 0, \epsilon)$ we get:

$$\frac{1}{\epsilon} \frac{\partial^2 g}{\partial x^2}(0, 0, \epsilon) + \frac{1}{\epsilon} \frac{\partial g}{\partial x \partial \epsilon}(0, 0, \epsilon) \frac{\partial \epsilon}{\partial x}(0, 0) = 0$$

So

$$\begin{aligned} \frac{\partial \epsilon}{\partial x}(0, 0) &= - \epsilon \left(\frac{\partial^2 g}{\partial x \partial \epsilon}(0, 0, \epsilon) \right)^{-1} \left(\frac{1}{\epsilon} \frac{\partial^2 g}{\partial x^2}(0, 0, \epsilon) \right) \\ &= - \left(\frac{\partial^2 g}{\partial x \partial \epsilon}(0, 0, \epsilon) \right)^{-1} \left(\frac{\partial^2 g}{\partial x^2}(0, 0, \epsilon) \right) \end{aligned}$$

which according condition(3)and(4)we can get

$$\frac{\partial \epsilon}{\partial x}(0, 0) \neq 0.$$

Then $(0, 0, \epsilon)$ is transcritical bifurcation point of singularly perturbed ODEs(2.2.1).

□

2.3.3 Pitchfork Bifurcation in Singularity perturbed ODEs

It is the transition from a stable equilibrium one into new stable equilibria two where separated by a saddle[42].

Theorem 2.3.3. Consider the system (2.3.2) and the extended system reduction (2.2.5) satisfy conditions of Theorem 2.2.2, if the following conditions hold:

1. $F_{xx}(0, 0, \epsilon) = 0$
2. $F_{\epsilon}(0, 0, \epsilon) = 0$
3. $F_{x\epsilon}(0, 0, \epsilon) \neq 0$
4. $F_{xxx}(0, 0, \epsilon) \neq 0$

Then $(0, 0, \epsilon) = (x_*, z_*, \epsilon_*)$ is pitchfork bifurcation point of the system (2.2.1) iff $(0, 0, \epsilon, \phi)$ is nonsingular equilibrium point of extended system reduction (2.2.5).

Proof. Let $(x_*, z_*, \epsilon_*) = (0, 0, \epsilon)$ is pitchfork bifurcation point of system (2.2.1). Then by Theorem 2.2.1 that $(x_*, z_*, \epsilon_*, \phi_*) = (0, 0, \epsilon, \phi)$ is equilibrium point of extended system reduction (2.2.5). According Theorem 2.2.2 we have to $(0, 0, \epsilon, \phi)$ is nonsingular of equilibrium point of extended system reduction (2.2.5). Now we will prove that $(0, 0, \epsilon)$ is pitchfork bifurcation point, under theorem conditions at the above.

we want to prove that:

$$\frac{\partial \epsilon}{\partial x}(0, 0, \epsilon) = 0$$

and

$$\frac{\partial^2 \epsilon}{\partial x^2}(0, 0, \epsilon) \neq 0$$

Since $F_{x\epsilon}(0, 0, \epsilon) \neq 0$ then by implicit function theorem we have $M(x, z, \epsilon(x, z)) = 0$, where M has define Theorem 2.4.2 and derive w.r.t x at $(0, 0, \epsilon)$

$$\frac{\partial M}{\partial x}(0, 0, \epsilon) + \frac{\partial M}{\partial \epsilon} \left(\frac{\partial \epsilon}{\partial x}(0, 0) \right) = 0$$

then we get:

$$\frac{\partial \epsilon}{\partial x}(0, 0) = -\left(\frac{\partial^2 f}{\partial x \partial \epsilon}(0, 0, \epsilon)\right)^{-1} \left(\frac{\partial^2 f}{\partial x^2}(0, 0, \epsilon)\right)$$

From condition(4)we have:

$$\frac{\partial \epsilon}{\partial x}(0, 0) = 0.$$

To prove that

$$\frac{\partial^2 \epsilon}{\partial x^2}(0, 0, \epsilon) \neq 0.$$

Differentiation $M(x, z, \epsilon(x, z)) = 0$ twice w.r.t x at $(0, 0, \epsilon)$ we get:

$$\frac{\partial^3 f}{\partial x^3}(0, 0, \epsilon) + \frac{\partial^2 f}{\partial x \partial \epsilon}(0, 0, \epsilon) \frac{\partial^2 \epsilon}{\partial x^2}(0, 0) = 0$$

So we have:

$$\frac{\partial^2 \epsilon}{\partial x^2}(0, 0) = -\left(\frac{\partial^2 f}{\partial x \partial \epsilon}(0, 0, \epsilon)\right)^{-1} \left(\frac{\partial^3 f}{\partial x^3}(0, 0, \epsilon)\right).$$

From condition(3)and(4)we get:

$$\frac{\partial^2 \epsilon}{\partial x^2}(0, 0) \neq 0.$$

By the same way, if we use $N(x, z, \epsilon) = 0$ and $\frac{1}{\epsilon}g(x, z, \epsilon) = 0$ we can prove the same result.

Then $(0, 0, \epsilon)$ is pitchfork bifurcation point of singularly perturbed ODEs(2.2.1).

□

Example 2.3.1. Consider the singularly perturbed ODEs:

$$\begin{aligned}\dot{x} &= xz \\ \epsilon \dot{z} &= -x^2 + \epsilon\end{aligned} \quad (x, z) \in R^2, \epsilon \in R.$$

where ϵ is a bifurcation parameter, $0 < \epsilon \ll 1$ and chosen the $\phi_0 = (\phi_1, \phi_2) = (1, 0)^\top$ and $L^\top$ chosen to be $L^\top = (L_1, L_2) = (1, 0)^\top$. Then by extended system reduction:

$$G(0, 0, \epsilon, \phi) = \begin{pmatrix} xz \\ -x^2 + \epsilon \\ z\phi_1 \\ -2x\phi_2 \\ x\phi_1 \\ 0 \\ L_1\phi_1 + L_2\phi_2 - 1 \end{pmatrix} = 0$$

$$F_\epsilon(0, 0, \epsilon) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0.$$

$$F_{xx}(0, 0, \epsilon) = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \neq 0.$$

Then according to Theorem 2.4.1 we have $(0, 0, \epsilon)$ is fold bifurcation of singularly perturbed ODEs. we can conclude from the Jacobian matrix that

$$DG(0, 0, \epsilon, 1) = \begin{pmatrix} z & x & 0 & 0 \\ -2x & 0 & 1 & 0 \\ 0 & 1 & 0 & z \\ -2 & 0 & 0 & -2x \\ 1 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^\top \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$(0, 0, \epsilon, 1)$ is nonsingular of equilibrium point of extended system reduction (2.2.5).

Chapter 3

Bifurcation of Solution in Differential Algebraic Equation by Extended System Reduction Method

3.1 Introduction

In this chapter we will work to reduce singularity perturbed ODEs to obtain the differential algebraic equations when perturbed parameter $\epsilon = 0$. Then by using the extended system reduction method the bifurcation theory will be applied on DAEs.

This will permit us to apply the classical bifurcation theory on singularly perturbed ODEs for new system (DAEs). So, this will be shown by giving sufficient conditions for saddle, pitchfork and transcritical bifurcation to be occurred in such system. We will investigate the sufficient conditions for bifurcation to be occurred in singularly perturbed ODEs when perturbed parameter $\epsilon = 0$. Then this implies to discuss the bifurcation of solution for DAEs.

In section two discussed the extended system reduction for differential algebraic equations when $\epsilon = 0$ will given.

In section three we apply bifurcation theory on DAEs when μ is bifurcation parameter.

Consider the singularity parameterized ODEs:

$$\begin{aligned}\dot{x} &= f(x, z, \mu, \epsilon) \\ \epsilon \dot{z} &= g(x, z, \mu, \epsilon),\end{aligned}$$

where $x \in R, z \in R, \mu, \epsilon \in R$,

when ϵ equal to zero we have the differential algebraic equations:

$$\begin{aligned}\dot{x} &= f(x, z, \mu, 0) \\ 0 &= g(x, z, \mu, 0),\end{aligned}$$

where $f : R \times R \times R \rightarrow R, g : R \times R \times R \rightarrow R$.

3.2 Extended System Reduction for DAE

Consider the parameterised DAEs:

$$\dot{x} = f(x, z, \mu, 0) \tag{3.2.1}$$

$$\begin{aligned}0 &= g(x, z, \mu, 0) \\ x \in R, z \in R, \mu \in R,\end{aligned} \tag{3.2.2}$$

where μ is bifurcation parameter and $\epsilon = 0$. Suppose that $\mathcal{C}$ is a set of all equilibrium points:

$$\mathcal{C} = \{(x_*, z_*, \mu_*, 0) \in R \times R \times R : f(x_*, z_*, \mu_*, 0) = g(x_*, z_*, \mu_*, 0) = 0\},$$

with rank condition:

$$\text{rank} g_z(x_*, z_*, \mu_*, 0) = 0.$$

Note that the equilibrium point of (3.2.2) should satisfy the constraint algebraic condition. Then the extended system reduction for the DAEs is given by:

$$G(x, z, \mu, \phi, 0) = \begin{pmatrix} f(x, z, \mu, 0) \\ g(x, z, \mu, 0) \\ g_z(x, z, \mu, 0)\phi \\ L^\top \phi - 1 \end{pmatrix} = 0 \quad (3.2.3)$$

And the Jacobian matrix of the reduced system is:

$$DG_y = \begin{pmatrix} f_x & f_z & f_\mu & 0 \\ g_x & g_z & g_\mu & 0 \\ g_{zx}\phi & g_{zz}\phi & g_{z\mu}\phi & g_z \\ 0 & 0 & 0 & L^\top \end{pmatrix} \quad (3.2.4)$$

where $y = (x, z, \mu, \phi, 0)$.

First, we associate a bifurcation points of DAEs (3.2.1)-(3.2.2) with equilibrium points of the extended system reduction (3.2.3) as in the following theorem.

Theorem 3.2.1. The point $(x_0, z_0, \mu_0, 0)$ is a bifurcation point of DAEs (3.2.1)-(3.2.2) iff the point $(x_0, z_0, \mu_0, \phi_0, 0)$ is an equilibrium point of extended system reduction (3.2.3).

Proof. Suppose that $(x_0, z_0, \mu_0, 0)$ is a bifurcation point of DAEs (3.2.1)-(3.2.2) then from rank condition we conclude that $Dg(x_0, z_0, \mu_0, 0)\phi_0 = 0$ for some eigenvector ϕ_0 and $L^\top \phi_0 - 1 = 0$ for some chosen vector $L^\top$. Hence $(x_0, z_0, \mu_0, \phi_0, 0)$ is an equilibrium point of extended system reduction (3.2.3), on the other side if $(x_0, z_0, \mu_0, \phi_0, 0)$ is an equilibrium point then $g(x_0, z_0, \mu_0, 0) = 0$ and rank condition is satisfied, this means that $(x_0, z_0, \mu_0, 0)$ is a bifurcation point of DAEs (3.2.1)-(3.2.2). $\square$

Now, we will show the bifurcation points of DAEs (3.2.1)-(3.2.2) are associated with the non-degenerated equilibrium points of extended system reduction (3.2.3), if certain non-degeneracy conditions on function g are satisfied.

Theorem 3.2.2. The jacobian matrix G_y is non singular if the following conditions hold:

$$f_\mu(x_0, z_0, \mu_0, 0) \notin \text{range}(g_x, g_z), \quad (3.2.5a)$$

$$g_\mu(x_0, z_0, \mu_0, 0) \notin \text{range}(g_x, g_z),$$

$$\text{range } f_x(x_0, z_0, \mu_0, 0) \cap \text{range } g_z(x_0, z_0, \mu_0, 0) = \{0\}, \quad (3.2.5b)$$

$$\ker f_x(x_0, z_0, \mu_0, 0) = \text{span}\{\phi\} \quad (3.2.5c)$$

$$\ker g_z(x_0, z_0, \mu_0, 0) = \text{span}\{\psi\} \neq \psi \in R,$$

$$g_{z0}\phi_0(x_0, z_0, \mu_0, 0) \notin \text{range } g_{x0}(x_0, z_0, \mu_0, 0), \quad (3.2.5d)$$

$$g_{z0}x_0(x_0, z_0, \mu_0, 0)\phi_0\psi_0 \notin \text{range } g_{z0}(x_0, z_0, \mu_0, 0). \quad (3.2.5e)$$

Proof. We shall prove that $\ker G_y^0 = \{0\}$ where G is extended system reduction of DAEs(3.2.1)-(3.2.2)

$$G_y^0 = \begin{pmatrix} f_x^0 & f_z^0 & f_\mu^0 & 0 \\ g_x^0 & g_z^0 & g_\mu^0 & 0 \\ g_{zx}^0\phi & g_{zz}^0\phi & g_{z\mu}^0\phi & g_z^0 \\ 0 & 0 & 0 & L^\top \end{pmatrix}$$

such that "0" denotes evaluation at $(x_0, z_0, \mu_0, \phi_0, 0)$. Let $(u, v, \lambda, \omega) \in \ker G_y$.

Then

$$G_y^0 \begin{bmatrix} u \\ v \\ \omega \\ \lambda \end{bmatrix} = \begin{pmatrix} f_x^0 u + f_z^0 v + f_\mu^0 \lambda \\ g_x^0 u + g_z^0 v + g_\mu^0 \lambda \\ g_{zx}^0 \phi u + g_{zz}^0 \phi v + g_{z\mu}^0 \lambda \phi + g_z^0 \omega \\ L^\top \omega \end{pmatrix}$$

According to the equation (3.2.5a) $\lambda = 0$ this given:

$$f_x^0 u + f_z^0 v = 0$$

$$g_x^0 u + g_z^0 v = 0$$

$$g_{zx}^0 u \phi + g_{zz}^0 v \phi + g_z^0 \omega = 0$$

$$L^\top \omega = 0$$

From (3.2.5b), (3.2.5c) we get that $f_z^0 v = 0, g_z^0 v = 0$ and $v = \alpha \phi, \alpha \in R$ and $\phi \neq 0$, so substituting into equations:

$$f_x^0 u + \alpha f_z^0 \phi = 0.$$

$$g_x^0 u + \alpha g_z^0 \phi = 0$$

$$g_{zx}^0 u \phi + \alpha g_{zz}^0 \phi \phi + g_z^0 \omega = 0$$

$$L^\top \omega = 0$$

According to (3.2.5d) and the equation $g_x^0 u + \alpha g_z^0 \phi = 0$ imply $\alpha = 0$, so $f_x^0 u = 0$ and $g_x^0 u = 0$ then from:

$$\ker f_x^0(x_0, z_0, \epsilon_0) = \text{span}\{\psi_0\}$$

we have $u = \beta \psi, \beta \in R, \psi \neq 0$ substituting into:

$$\beta f_x^0 \psi = 0$$

$$\beta g_x^0 \psi = 0$$

$$\beta g_{zx}^0 \phi \psi + g_z^0 \omega = 0.$$

$$L^\top \omega = 0.$$

Then according to (3.2.5c), $\beta = 0$ then

$$g_z^0 \omega = 0$$

$$L^\top \omega = 0.$$

According to condition:

$$\ker g_z^0(x_0, z_0, \epsilon_0) = \{\text{span}\{\phi\}\}$$

so $\omega = \gamma \phi, \gamma \in R$ and $\phi \neq 0$ this given

$$\gamma g_z^0 \phi = 0$$

$$\gamma L^\top \phi = 0.$$

this imply $\gamma = 0$ lead to $\omega = 0$. □

3.3 Bifurcation of Solution of DAEs by Extended System Reduction Method

In this section, after obtaining the differential algebraic equation we will apply the bifurcation theory again by taking the parameter ϵ equal to zero and μ is the bifurcation parameter for the algebraic part at equation (3.2.2). At bifurcation point, at least one eigenvalue of the Jacobian matrix g_z has zero real part.

We consider the following system of equations:

$$f(x_*, z_*, \mu_*, 0) = 0 \quad (3.3.1)$$

$$\begin{aligned} g(x_*, z_*, \mu_*, 0) &= 0 \\ x \in R, z \in R, \mu &\in R \end{aligned} \quad (3.3.2)$$

where $(x_*, z_*, \mu_*, 0) \in \mathcal{C}$ is singular equilibrium point.

The Jacobian matrix of the extended system reduction is nonsingular at equilibrium point when $\epsilon = 0$ is given by:

$$DG_y^* = \begin{pmatrix} f_x^* & f_z^* & f_\mu^* \\ 0 & & \\ g_x^* & g_z^* & g_\mu^* \\ 0 & & \\ g_{zx}^* & g_{zz}^* & g_{z\mu}^* \\ g_z^* & & \\ 0 & 0 & 0 \\ L^\top \end{pmatrix}$$

where “*” is superscript indicate evaluated at $y = (x_*, z_*, \mu_*, \phi_*, 0)$. The previous Theorem 3.2.1 showed under conditions that the Jacobian matrix is nonsingular imply $\ker G_y^* = \{0\}$.

In the following subsections bifurcation theory on DAEs when μ is bifurcation parameter and $\epsilon = 0$.

3.3.1 Fold(Saddle-node) Bifurcation of DAEs

The most sufficient conditions for the saddle-node bifurcation to the occurrence are:

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0$$

and

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0, 0, 0) \neq 0.$$

Theorem 3.3.1. Consider the system (3.3.2) and the extended system reduction (3.2.3). If the following conditions hold:

1. $g_x(0, 0, 0, 0) = 0$
2. $g_\mu(0, 0, 0, 0) \neq 0$.
3. $g_{xx}(0, 0, 0, 0) \neq 0$.

Then $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is fold bifurcation point of DAEs (3.2.1)-(3.2.2) iff $(0, 0, 0, \phi_*, 0)$ is non singular equilibrium point of extended system reduction (3.2.3).

Proof. If $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is fold bifurcation point of system (3.2.1)-(3.2.2), then by Theorem 3.2.1 that $(x_*, z_*, \mu_*, \phi_*, 0) = (0, 0, 0, \phi, 0)$ is equilibrium point of extended system reduction (3.2.3). From Theorem 3.2.2 we get $\ker G_y^*(0, 0, 0, \phi, 0) = 0$ this implies that $(0, 0, 0, \phi_*, 0)$ is non degenerate equilibrium point of the extended system reduction.

Suppose $(0, 0, 0, \phi_*, 0)$ non singular equilibrium point of extended system reduction (3.2.3).

Then from the non hypobulic conditions (3.2.3) we have $g(x_*, z_*, \mu_*, 0) = 0$ and $g_z(x_*, z_*, \mu_*, 0)\phi = 0$, from this we conclude that $(0, 0, 0, 0)$ is bifurcation point.

Now we will prove that $(0, 0, 0, 0)$ is fold bifurcation point, under the conditions given in the theorem above. We want to prove that

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0$$

and

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0, 0, 0) \neq 0$$

By implicit function theorem to equation (3.2.2) we have:

$$g(x, z, \mu(x, z), 0) = 0 \quad (3.3.3)$$

Differentiating the equation (3.3.3) w.r.t x at $(0, 0, 0, 0)$ we get:

$$\frac{\partial g}{\partial x}(0, 0, 0, 0) + \frac{\partial g}{\partial \mu} \frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0$$

and

$$\frac{\partial \mu}{\partial x}(0, 0) = -\left(\frac{\partial g}{\partial \mu}(0, 0, 0, 0)\right)^{-1} \left(\frac{\partial g}{\partial x}(0, 0, 0, 0)\right) = 0.$$

Then from condition (1) and (2) we have:

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0.$$

Now differentiate equation $g(x, z, \mu(x, z), 0) = 0$ twice w.r.t x at $(0, 0, 0, 0)$ and use the property $\frac{\partial \mu}{\partial x} = 0$ we get:

$$\frac{\partial^2 g}{\partial x^2}(0, 0, 0, 0) + \frac{\partial g}{\partial \mu}(0, 0, 0, 0) \left(\frac{\partial^2 \mu}{\partial x^2}(0, 0)\right) = 0.$$

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0) = -\left(\frac{\partial g}{\partial \mu}(0, 0, 0, 0)\right)^{-1} \left(\frac{\partial^2 g}{\partial x^2}(0, 0, 0, 0)\right).$$

from condition (1) and (2) we can get:

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0) \neq 0.$$

Then $(0, 0, 0, 0)$ is fold bifurcation point of DAEs (3.2.1)-(3.2.2) □

3.3.2 Transcritical Bifurcation of DAEs

The most sufficient condition for the transcritical bifurcation to be occurred in DAEs, which are given for equilibrium points that satisfy:

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) \neq 0$$

We define that:

$$\begin{aligned} f(x, z, \mu, 0) &= xV(x, z, \mu, 0) \\ g(x, z, \mu, 0) &= xU(x, z, \mu, 0) \end{aligned}$$

where $f : R \times R \times R \longrightarrow R$, $g : R \times R \times R \longrightarrow R$ and μ is a bifurcation parameter $\epsilon = 0$. And $f(0, z, 0, 0) = \frac{\partial f}{\partial x}$ at $x = 0$ and $g(0, z, 0, 0) = \frac{\partial g}{\partial x}$ at $x = 0$.

The following theorem is related to transcritical bifurcation of DAEs:

Theorem 3.3.2. Consider the system (3.3.1)-(3.3.2) and the extended system reduction (3.2.3). If the following conditions hold:

1. $g_\mu(0, 0, 0, 0) = 0$.
2. $g_{x\mu}(0, 0, 0, 0) \neq 0$.
3. $g_{xx}(0, 0, 0, 0) \neq 0$.

Then $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is trans critical bifurcation point of DAEs (3.2.1)-(3.2.2) iff $(0, 0, 0, \phi_*, 0)$ is non singular equilibrium point of extended system reduction (3.2.3).

Proof. Let $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is transcritical bifurcation point of DAEs (3.2.1)-(3.2.2). From Theorem 3.3 we have to $(0, 0, 0, \phi, 0)$ is non singular equilibrium point of extended system reduction (3.2.3).

Now we will prove that $(0, 0, 0, 0)$ is transcritical bifurcation point, under the conditions given in the theorem above. We want to prove that

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) \neq 0.$$

Since $g_{x\mu}(0, 0, 0, 0) \neq 0$ then $U_\mu(0, 0, 0, 0) \neq 0$, so by implicit function theorem we have $U(x, z, \mu(x, z)) = 0$, and differentiating w.r.t x we get:

$$\frac{\partial U}{\partial x}(x_*, z_*, \mu_*, 0) + \frac{\partial U}{\partial z}(x_*, z_*, \mu_*) \frac{\partial z}{\partial x} + \frac{\partial U}{\partial \mu} \left(\frac{\partial \mu}{\partial x}(x_*, z_*) + \frac{\partial \mu}{\partial z}(x_*, z_*) \frac{\partial z}{\partial x} \right) = 0$$

at $(0, 0, 0, 0)$ we have:

$$\frac{\partial^2 g}{\partial x^2}(0, 0, 0, 0) + \frac{\partial^2 g}{\partial x \partial \mu}(0, 0, 0, 0) \frac{\partial \mu}{\partial x}(0, 0) = 0,$$

and

$$\frac{\partial \mu}{\partial x}(0, 0) = -\left(\frac{\partial^2 g}{\partial x \partial \mu}(0, 0, 0, 0)\right)^{-1} \left(\frac{\partial^2 g}{\partial x^2}(0, 0, 0, 0)\right)$$

From condition(2) and (3) we have

$$\frac{\partial \mu}{\partial x} \neq 0$$

. Then $(0, 0, 0, 0)$ is transcritical bifurcation point of DAEs (3.2.1)-(3.2.2). $\square$

3.3.3 Pitchfork Bifurcation of DAEs

The most sufficient condition for the pitchfork bifurcation to be occur in DAEs, which are given for equilibrium points satisfy that:

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0$$

and

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0, 0, 0) \neq 0.$$

Theorem 3.3.3. Consider the system (3.3.2) and the extended system reduction (3.2.3). If the following conditions hold:

1. $g_\mu(0, 0, 0, 0) = 0, g_{xx}(0, 0, 0, 0) = 0.$
2. $g_{x\mu}(0, 0, 0, 0) \neq 0.$
3. $g_{xxx}(0, 0, 0, 0) \neq 0.$

Then $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is pitch bifurcation point of DAEs (3.2.1)-(3.2.2) iff $(0, 0, 0, \phi_*, 0)$ is non singular equilibrium point of extended system reduction (3.2.3).

Proof. Let $(x_*, z_*, \mu_*, 0) = (0, 0, 0, 0)$ is pitchfork bifurcation point of DAEs (3.2.1)-(3.2.2). From Theorem 3.2.2 we have to $(0, 0, 0, \phi, 0)$ is non singular of equilibrium point of extended system reduction (3.2.3). Now we will prove that $(0, 0, 0, 0)$ is pitchfork bifurcation point, under the conditions given in the theorem above.

We want to prove that:

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0$$

and

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0, 0, 0) \neq 0$$

Since $g_{x\mu}(0, 0, 0, 0) \neq 0$ then by implicit function theorem we have $U(x, z, \mu(x, z)) = 0$, where U has define Theorem 3.2.4.

Differentiate w.r.t x at $(0, 0, 0, 0)$:

$$\frac{\partial U}{\partial x}(0, 0, 0, 0) + \frac{\partial U}{\partial \mu} \left(\frac{\partial \mu}{\partial x}(0, 0) \right) = 0$$

then

$$\frac{\partial \mu}{\partial x}(0, 0) = - \left(\frac{\partial^2 g}{\partial x \partial \mu}(0, 0, 0, 0) \right)^{-1} \left(\frac{\partial^2 g}{\partial x^2}(0, 0, 0, 0) \right)$$

From condition 1 and 2 we have

$$\frac{\partial \mu}{\partial x}(0, 0, 0, 0) = 0.$$

Now we prove that $\frac{\partial^2 \mu}{\partial x^2}(0, 0, 0, 0) \neq 0$ Differentiate the equation $U(x, z, \mu(x, z)) = 0$ twice w.r.t x at $(0, 0, 0, 0)$ we can get:

$$\frac{\partial^2 U}{\partial x^2}(0, 0, 0, 0) + \frac{\partial^2 U}{\partial \mu \partial x}(0, 0, 0, 0) + \frac{\partial \mu}{\partial x}(0, 0) \frac{\partial U}{\partial \mu}(0, 0, 0, 0) \left(\frac{\partial^2 \mu}{\partial x^2}(0, 0) \right) = 0.$$

From the property $\frac{\partial \mu}{\partial x}(0, 0) = 0$ then we get:

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0) = - \left(\frac{\partial^2 g}{\partial x \partial \mu}(0, 0, 0, 0) \right)^{-1} \left(\frac{\partial^3 g}{\partial x^3}(0, 0, 0, 0) \right)$$

According to condition (2) and (3) we get:

$$\frac{\partial^2 \mu}{\partial x^2}(0, 0) \neq 0.$$

Then $(0, 0, 0, 0)$ is pitchfork bifurcation point of DAEs (3.2.1)-(3.2.2). $\square$

Example 3.3.1. Consider the system

$$\begin{aligned}\dot{x} &= \epsilon x - z \\ \epsilon \dot{z} &= \mu - x^2 - \epsilon z\end{aligned}, \quad x \in R^2, z \in R^2, \mu, \epsilon \in R.$$

when $\epsilon = 0$, μ is bifurcation parameter and chosen to be $L^\top \phi = 1$ we get:

$$\begin{aligned}\dot{x} &= -z \\ 0 &= \mu - x^2\end{aligned}, \quad x \in R, z \in R, \mu, \epsilon \in R$$

So the extended system reduction as:

$$G(0, 0, 0, \phi, 0) = \begin{pmatrix} -z \\ \mu - x^2 \\ -2x\phi \\ l^\top \phi - 1 \end{pmatrix}$$

$$g_\mu(0, 0, 0, 0) = 1 \neq 0$$

$$g_{xx}(0, 0, 0, 0) = -2 \neq 0$$

Then according to Theorem 3.3.1 we have $(0, 0, 0, 0)$ is fold bifurcation of DAEs. we can conclude from the Jacobian matrix that

$$DG(0, 0, 0, 1, 0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -2x & 0 & 1 & 0 \\ -2 & 0 & 0 & -2x \\ 0 & 0 & 0 & L^\top \end{bmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$(0, 0, 0, 1, 0)$ is nonsingular equilibrium point of extended system reduction (3.2.3).

Chapter 4

Conclusion and Future Works

Conclusion

In this thesis we have shown that the differential algebraic equation (DAEs) can be studied through the singularly perturbed system by extended system reduction method.

That is the classical theorems of bifurcation that can be applied to singularly perturbed(ODEs). The known kinds of bifurcation, fold(saddle node), Trans critical and pitchfork bifurcation have been investigated for (ODEs) in the case when $0 < \epsilon \ll 1$.

Also the case when $\epsilon = 0$ was carefully dealt with as to show that bifurcation occurs in DAEs by using extended system reduction method and Implicit Function Theorem.

Future Work

This work can be generalized by looking for the following subjects:

- Study Hopf bifurcation of solution in singularly ODEs by extended system reduction.
- Study Hopf bifurcation in differential algebraic equations DAEs.
- Study dabble bifurcation and give the sufficient condition for this kind of bifurcation to be occurred in singularly perturbed ODEs and by considering the case when ϵ equal to zero, the study can be applied on DAEs.

List of Publications

- (1) Ayat S. and Kamal H. Bifurcation of Solution in Singularly perturbed Equations by Extended System Reduction ,(2018).Journal of College of Education for Pure Science(JCEPS), Iraq.
- (2) Ayat S. and Kamal H. Bifurcation of Solution in Differential Algebraic Equations by Extended System Reduction,2018,submitted.

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