

**Blow up method for studying bifurcation
of solution in singular perturbation
ordinary differential equation and
differential algebraic equation**

A Thesis Submitted to

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by

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To My Parents and My Brothers.

Abstract

This thesis deals with studying a new subject of a singularity perturbed ordinary differential equations system. It is considered the foundation base to get a differential algebraic equations, where the concept of singularity perturbed ODEs is explained. Then it studies the ways to deal with the perturbation parameter ϵ through two case studies:

The first case: Is the study of behavior of solution for singularity perturbed ODEs when perturbation parameter $0 < \epsilon \leq 1$. The above equation has been reduced by using (The implicit function theorem) for this purpose, we obtain an equation involving the central manifold and thus, applicate (Blow up method). Moreover, for this purpose of obtaining a solution curve for the system (Fast-Slow system).

The second case: Is the study of behavior for perturbation parameter ϵ when it approaches to zero. In this case we get a differential algebraic equations system that explains the relationship between the two systems. Then the bifurcation theory is applied on the last system according to singularity perturbed ODEs. In addition, sufficient conditions for the occurrence of some types of bifurcation in the solution are given, such as (Fold, Pitchfork, Transcritical and Hopf Bifurcation). Depending on the proof of theories to reduce the differential algebraic equation into ordinary differential equation using the (Implicit Function Theorem). For this purpose, proof of bifurcation that oc-

curs in differential algebraic equation in this kind of situations is depended on the nature and behavior of the solution at the level of each state of bifurcation.

Keywords: Blow up Method, Bifurcation Theory, Singular perturbation system Differential-Algebraic Equations

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I certify that this thesis (**Blow up method for studying bifurcation of solution in singular perturbation Ordinary differential equations and differential algebraic equations**) was prepared under my supervision at departement of mathematics/ College of education for pure Science / University of Thi-Qar, as a partial fulfilment of requirements for degree of master of science (*M.Sc*) in mathematics.

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List of Symbols

$\dot{x}$	Derivative of variable x with respect to the time t .
x_*	The equilibrium point.
R	The set of real number.
R^n	The set of real valued vectors with n dimension.
A^{-1}	The inverse of A .
I	Closed interval of real numbers set.
$\det A$	The determined of the matrix A .
$\ \cdot\ $	The Euclidean norm.
J	The Jacobian matrix.
J_r	The reduced Jacobian matrix.
$\lim$	A limit of a function.
ϵ	A perturbation parameter.
C^r	The normed vectorspace of r times differentiably continuous real functions.
D_f	The derivative of the function f .
DAE	A differential algebraic equation.
ODE	An ordinary differential equation.
SP	Singular perturbation system.
S^2	Asphere

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Chapter 1

Introduction

1.1 General Review

The importance of mathematics in the study of problems arising from the phenomena, and the increasing success with which it has been used to model situations ranging from the purely deterministic to the stochastic, is well established.

This thesis consists of two parts of the study:

1. Application of blow up method to singular perturbation ordinary differential equation and differential algebraic equation.
2. Application of bifurcation theory to singular perturbation ordinary differential equation and differential algebraic equation by using blow up method.

Earlier, the study of the dynamical systems was almost certainly on looking for solutions and the methods to find these solution. At the end of the eighteenth century (1800), the new studies of dynamical system by Poincare began. The present study is the dynamical

behavior of physical processes without the need to find its solutions. Presently this study, is called the qualitative study of dynamical systems. Then concerning this study new cases are raised, such as vector fields, flows, trajectories, topological equivalence, etc. The field of differential equations goes back to the time of Newton and Leibniz. Leibniz invented calculus because he realized that many of the laws of nature are governed by what we now call differential equations [17].

The Implicit function theorem is important in this work as it will be seen. So it is necessary to explain it.

An autonomous first order ordinary differential equation is any equation of the form:

$$\frac{dx}{dt} = f(x), \tag{1.1.1}$$

where f is a vector values function which can be nonlinear. In the context of differential equations, autonomous means that the derivative can be expressed without any explicitly reference to time t . Ordinary differential equation is a very powerful tool to describe dynamical systems. A nonlinear system is a system in which the output is not directly proportional to the input.

Typically, the behavior of a nonlinear system is described in mathematics by a nonlinear system of equations, which is a set of simultaneous equations in which the unknown (or the unknown functions in the case of differential equations) appear as variables of a polynomial of degree higher than one or in the argument of a function which is not a polynomial of degree one.

Dealing with non linear equations is more difficult. Let us discuss the non-linear in [47]

$$\dot{x} = \sin(x).$$

The separation of variables leads to:

$$\frac{dx}{\sin(x)} = \csc(x)dx = dt,$$

which can be integrated with the result:

$$t = \ln\left(\frac{\csc(x_*) + \cot(x_*)}{\csc(x) + \cot(x)}\right).$$

Even in this simple non-linear example, the behavior of the system is not easy to understand from this solution but graphical analysis shows the most important properties.

Plotting a phase portrait or vector field in figure (1.1), stable and unstable fixed points can be determined[47].

The systems dynamics in (1.1.1) corresponds to flow on the line.

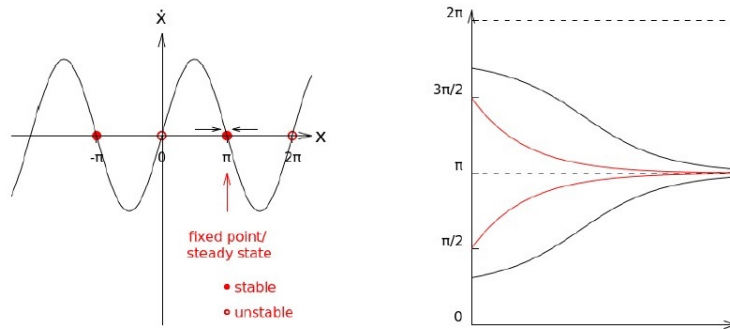


Figure 1.1: Stable and Unstable Fixed Point

To study the behavior of non linear system near a fixed point, we can linearize the system about fixed point x_* of $\dot{x} = f(x) = 0$. Then by determining the eigenvalues, we can describe the stability of the system around x_* , where

1. If $\dot{f}(x_*) < 0$ then x_* , stable fixed point, exponential decay.
2. If $\dot{f}(x_*) > 0$ then x_* , unstable fixed point.
3. If $\dot{f}(x_*) = 0$ then investigations are needed.

Bifurcation theory is the study of equation with multiple solutions and the mechanism of equilibrium is points when it is created and destroyed as the parameter varies. One and two dimensional projections are the most important. They fall under the category of problems called bifurcation at a simple eigenvalue.

To get an idea of what this topic is all about, let's go back to elementary algebra and look at the quadratic function $f(x) = x^2 + x + c$, where c is a constant. We should realize that the zeros of this function depends on the parameter c . To see this , let's consider the following example taken from [42]:

$$x^2 + x + c = (x + \frac{1}{2})^2 + (c - \frac{1}{4}). \quad (1.1.2)$$

Clearly, the item $(x + \frac{1}{2})^2$ is always nonnegative, so that if $c > \frac{1}{4}$, the expression in (1.1.2) is always greater than zero, and the quadratic equation $x^2 + x + c = 0$ has no real solution. If $c = \frac{1}{4}$, then the only solution is $x = -\frac{1}{2}$. Finally, if $c < \frac{1}{4}$, then it has two solutions $x = -\frac{1}{2} \mp \sqrt{\frac{1}{4} - c}$.

The important point in this example is that $\frac{1}{4}$ is the value of the parameter c at which the nature of the solutions of the quadratic equation changes. We say that $c = \frac{1}{4}$ is a bifurcation value because as c decreases through $\frac{1}{4}$, the solution $x = 0$ splits into two solutions and $(\frac{1}{4}, 0)$ is a bifurcation point. Now take figure(1.3) to explain this idea [23].

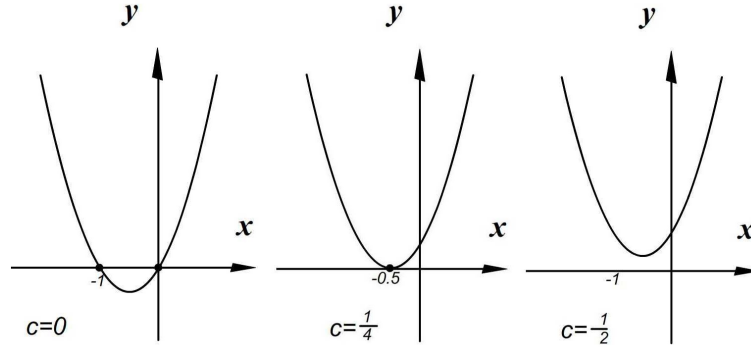


Figure 1.2: Behavior of $f(x) = x^2 + x + c$ when c changed.

As a result, such nonlinear phenomena are best understood as some forms of perturbation of their linear approximations. The aim of perturbation theories is to determine the behavior of the solution of equation depending on a small real valued parameter that is simplified in some way when $\epsilon = 0$ as the form $f_\epsilon(x) = 0$.

Perturbation theory was created by Rayleigh and Schrodinger. Rayleigh gives a formula for computing the natural frequencies and modes of a vibrating system deviating slightly from a simpler system which admits a complete determination of the frequencies and modes. Mathematically speaking, the method is equivalent to an approximate solution of the eigenvalue problem for a linear operator slightly different from a simpler operator for which the problem is completely solved.

Schrodinger developed a similar method, with more generality and systematization, for the eigenvalue problems that appear in quantum mechanics[33].

Perturbation theory is called regular if the perturbed problem for small, nonzero values of ϵ is qualitatively the same as the unperturbed problem for $\epsilon = 0$. Regular pertur-

bation obtains a convergent expansion of the solution with respect to ϵ , consisting of the unperturbed solution and higher order corrections and obtains an asymptotic, but possibly divergent, expansion of the solution, which depends singularly on the parameter ϵ [22] but it is called singular if it is not regular.

The analytical technique of singular perturbation theory has a long history. In recent years it has been used in many different ways, and its importance which has been enhanced is, by it having been used in various fields to derive sequences of asymptotic approximations. Each is with a higher order of accuracy than its predecessor. Mathematical problems that make extensive use of a small parameter were probably first described by J. H. Poincare (1854 –1912) as part of his investigations in celestial mechanics.

New analytical approaches have stimulated the development of numerical techniques for the solution of this major class of practical problems. In (1996) [10] studied singular perturbation problems, depending on a small parameter such that the solutions show a nonuniform behavior as the parameter tends to zero.

Singular perturbation theory, similar to reduction methods that can be seen as the limiting system as some dynamics become arbitrarily fast which was studied in [34] in (1986). It has many advantages in the applications of real physical systems such as aircraft, power systems, nuclear reactors, chemical reactors, electrical circuits, induction motors, robots, large space flexible structures, synchronous machines, cars, and others. Singular perturbations and time-scale techniques were introduced to control engineering in the late (1960s). They become common tools for the modeling, analysis and design of control systems. In the control literature, the singular perturbation approach of Tikhonov 1952 and Vasileva 1963 was first applied to optimal control and regulator

design by Kokotovic and Sannuti 1968 and Sannuti and Kokotovic 1969 [34].

The dynamic behavior of physical processes is usually modeled via differential equations but if the states of the physical system are in some ways constrained, for example by conservation laws such as Kirchhoffs laws in electrical networks, or by position constraints such as the movement of mass points on a surface, then the mathematical model will also contain algebraic equations to describe these constraints. Such systems, consisting of both differential and algebraic equations are called differential algebraic systems, algebraic differential systems, implicit differential equations or singular systems.

The origins of DAE theory can be traced back to the work of K. Weierstrass and L.Kronecker on parameterized families of bilinear forms[23]. In terms of matrices, these pencils are applied to the analysis of linear systems of ordinary differential equations with a possibly singular leading coefficient matrix by F. R. Gantmacher[43]. The key ideas support what nowadays is known as the differentiation index of a semi-explicit DAE.

The study of the topological behavior of the solution of a real planar vector field

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y),$$

in a neighborhood of a singular point is one of the main unsolved problems in the qualitative theory of differential systems. The main technique to perform the desingularization of linearly zero singular points is the blowup method. The main geometric idea of the method arise in algebraic geometry in the context of desingularization of algebraic varieties [24], where one replaces certain singular points by projective space. The resulting variety either has no singular points anymore or one can try to repeat the blow up.

In the context of ODEs, the classical strategy involves using a spherical blow up as one works in real space and not in the context of (complex) projective space. The key difference to the algebraic geometry blow up is that one also has to keep track of the dynamics on the blown up space. There has been a tremendous amount of work on using the blow up technique for planar ODEs [48], [11], canard solutions [13], [31] and a large variety of other problems in the theory of multiple time scale dynamical systems [32]. In [30] the sphere is replaced by hyperbolic space, i.e. by using a manifold with constant negative curvature.

The desingularization theorem for planar vector fields was first stated by Bendixson in 1901. The desingularization procedure is extended to C^∞ vector fields of Lojasiewicz type in [11]. The quasi homogeneous blow up is a widely used extension of the blow up technique. It was already used by Lyapunov but it was essentially put forward as a systematic and more powerful technique in and especially. The desingularization procedure using quasi homogeneous blow ups was also applied in [12].

1.2 Singular Perturbation System

The study of systems of non-linear differential equations which is in general rather difficult and an analytical description of the phase space dynamics is usually impossible. For different classes of non linear ODEs methods are developed that can give an insight on the behavior of the system. One of these classes is related to singular perturbation problems. Which can be written in the form in [3]:

$$\epsilon \dot{x} = f(x, y, \epsilon), \tag{1.2.1}$$

$$\dot{y} = g(x, y, \epsilon),$$

with $x \in R^n$, $y \in R^m$, $\epsilon \in R$, $0 < \epsilon \ll 1$, $f : R^{n+m+1} \longrightarrow R^n$ $g : R^{n+m+1} \longrightarrow R^m$ and $f, g \in C^r$, $r < \infty$. The dot is the derivative with respect to the time τ . Since ϵ is small, in times of order of τ the variable x evolves slower than y .

With Perturbation methods, they generally try to write solutions of equations as power series in a small parameter. To understand the steady state approximation, we find that we are led to problems in which the small parameter multiplies a derivative. Field with active research is going on these days. These problems are called singular perturbation in order to distinguish them from the regular perturbation problems.

When the $O(\epsilon)$ terms as perturbations are considered indeed, the approximation of $\epsilon = 0$ in (1.2.1) leads to a system which has a different behavior from the perturbed one. System (1.2.1) is also called slow fast, because the set of differential equations is split into two parts which evolve in two different time scales. In equation (1.2.1) the first m equations evolve slower than the following n . The measure of the separation between the two time scales is given by the parameter $\epsilon \ll 1$. The class of singular perturbation problems is rather important and it is common to find physical examples belonging to it.

1.3 Previous Works

After the pioneering work of Dumortier and Roussarie [13], it turned out that the blow up method is a powerful tool in the analysis of singular perturbation problems with non hyperbolic points. By now this method finds numerous applications in the analysis of the dynamics associated with non hyperbolic points of singularly perturbed differential equations [13]. The blow up method is used to desingularize such singularities based

on certain coordinate transformations. The most simple case $g(0, 0) \neq 0$ in (1.2.1) is deeply investigated by Krupa and Szmolyan, [31] with the aid of a geometric view point. Straight forward extensions to higher dimensional cases are done by Szmolyan and Wechselberger [45] and by Mishchenko and Rozov [39] but if $g(0, 0) = 0$, in (1.2.1) then more complex phenomena such as canard explosion can occur if the dynamics (1.1.2) has fixed points on a set of fold points. Such situations are investigated by [15, 13, 31] by using the blow up method.

In many of these applications, degenerate points are blown up to spheres, however there are also works where higher dimensional degenerate objects are blown up, for example non hyperbolic curves are blow up to cylinders [21]. Also in [21], additional blow ups and used the points are treated, if a certain blow up leads to a less degenerate problem which however still has degenerate points.

In [32] the cases that fast systems undergo a transcritical bifurcation and a pitchfork bifurcation are studied by blow up method.

Bifurcation in DAE is not easy to study. Authors try to find a system of equations which is in some ways equivalent to the origin at DAE system. Ian Dobson in 1992 [18] depended only on the constraint. Kamal . H . Yasir Al-yassiri [53] in (2007) worked on the stability of Hessenbreg DAEs for index 2 and 3. In (2013), Ibrahim Hussien studied Bifurcation in DAEs without reduction with application on circuit simulation [23]. Hassan Shather in (2015) worked on the stability on DAEs by Floquet Theory.

Stability issue was been studied first by applying singular ODEs like for example author in [28] in 1983, then generalized for DAEs depending on DAE system with regular matrix pencil [54] in 1997. Author in [37] studied this issue depending on Lyapunov function in 2012.

1.4 Our Work

The work in [29] introduces blow up method on ordinary differential equation to study the phase portrait of solution near singular point. In this thesis, we generalize the work in [29] by using the blow up method to study the bifurcation of solution near the singular point. The method is applied on the perturbation ordinary differential equation and differential algebraic equation. Then by introducing the time-scale of singularity perturbed ODEs is introduced, and studying the cases of perturbation parameter are studied. The first case is that when perturbation parameter ϵ approaches to zero, a differential algebraic equations system (DAEs) will be obtained, then the study of bifurcation theory on differential algebraic equations system (DAEs) will be given. The second case studies the problem when the perturbation parameter is $0 < \epsilon \ll 1$ then we use polar blow up method to investigate the bifurcation in singularity perturbed ODEs. The two cases will be compared according to the procedure of the reducing model, the bifurcation theory conditions, and the normal form of the bifurcation diagram.

Construction of this thesis

The important of the present topic under investigation is the study of singularity perturbed ODEs that represents a basic source for originating differential algebraic equation (DAEs).

Chapter 2, The second introduce basic definitions and important ideas that are required in the present study.

Chapter 3, The fourth states the second case is by applying the implicit function

theorem to obtain reduced model from singularity perturbed ODEs when perturbation parameter is greater than zero (i.e not equal to zero) and bifurcation theory on ODEs is studied by using polar blow up method.

Chapter 4, The third studies bifurcation theory on reduced problem from singularity perturbed ODEs when perturbation parameter is equal to zero , depending on directional blow up method and implicit function theorem.

Chapter 5, The fifth chapter includes the Conclusion and future work of the present topic.

Chapter 2

Concepts and Definitions

Some of the important definitions and concepts of differential algebraic equations are introduced. Systems of differential algebraic equations sometimes take different forms according to the nature of the model structure.

2.1 Differential Algebraic Equations

Differential algebraic equations (DAEs) present the state of the art in mathematical modeling of dynamical systems approximation in all areas of science and engineering now days. The topic is well studied from an analytic, numerical and control theoretical point of view and several monographs are available that cover different aspects of the subject.

The mathematical model can usually be written in the form:

$$F(x, \dot{y}, t) = 0, \tag{2.1.1}$$

where $F : R^n \times R^m \times I \longrightarrow R^K$ is sufficiently smooth. If $F_{\dot{y}}$ is nonsingular, it is possible to solve DAEs formally above for $\dot{y}$ in order to obtain an ordinary differential equation. However, if $F_{\dot{y}}$ is singular, this is no longer possible and the solution $x(t)$ has to satisfy certain algebraic constraints [44].

2.2 Classification of DAE

1 Nonlinear DAEs[4]

The general form of DAE is :

$$F(x, \dot{y}, t) = 0, \quad (2.2.1)$$

it is called fully implicit form. This form is nonlinear if F is nonlinear w.r.t any one of $t, x, \dot{y}$.

2 Linear DAEs[43]

The form is

$$F(x, \dot{y}, t) = A(t)\dot{y} + B(t)x + C(t) = 0. \quad (2.2.2)$$

It is called linear DAEs. The DAEs in the form (2.2.2), is called linear time invariance (*LTI*), when $B(t)$ is constant matrix for all $t \in R$, otherwise, it is linear time-varying (*LTV*) and it is called DAE with constant coefficients when $A(t)$ and $B(t)$ are constant for all $t \in R$.

3 Semi explicit DAEs[43]

A differential algebraic equation (2.2.1) can be written as:

$$\dot{x} = f(t, x, y) \quad (2.2.3)$$

$$0 = g(t, x, y).$$

It is called semi explicit DAE form, where x represents differential variable and y is called an algebraic variable. So the term differential algebraic equation is sometimes used in the literature to mean the semi explicit equation (2.2.3). DAEs of this form can also be described as an ODE with constraint.

4 Hessenberg DAEs[43]

The DAEs (2.2.1) is in **Hessenberg form** of size m and written as:

$$\begin{aligned} \dot{x}_1 &= f_1(x_0, x_1, \dots, x_m, t) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_{m-1}, t) \\ &\vdots \\ \dot{x}_i &= f_i(x_{i-1}, x_i, \dots, x_{m-i}, t) \\ &\vdots \\ 0 &= f_m(x_{m-1}, t), \end{aligned}$$

where $i=\{1, \dots, m\}$ and the matrix $\frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \frac{\partial f_1}{\partial x_3}$ is invertible in a neighborhood of the solution. It is assume that f_i is sufficiently smooth. Then the above Hessenberg system of size m is a DAE of index m and is solvable (it has at least one solution).

5 Quasilinear DAEs[43]

Quasi linear autonomous DAEs are systems of the form:

$$A(x)\dot{x} = f(x),$$

where $A : W_0 \rightarrow R^{m \times m}$ and $f : W_0 \rightarrow R^m$ are sufficiently smooth mappings respectively and $W_0 \subseteq R^m$ is an open set. Many cases of application deal with $A(x)$ as a singular matrix.

6 Semilinear DAEs[43]

Semilinear equations comprise in particular the semi-explicit DAEs (2.2.3), since the latter can be written as:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(y, z) \\ g(y, z) \end{bmatrix},$$

which is a system of the form

$$A\dot{x} = h(x),$$

with $x = (y, z)$, $h = (f, g)$. Conversely, semi linear DAEs may be written in a semi-explicit form.

2.3 Index of DAE

Different approaches of classifying the difficulties that arise in DAEs lead to different so-called index concepts, where the index is a 'measure of difficulty' in the analytical or numerical treatment of the DAE. In this contribution, the major index concepts will be surveyed and put in perspective with each other as far as this is possible.

Since most index concepts are only defined for uniquely solvable square systems with $m = n$, this case is only studied here. The higher the index is, the more difficult it is to solve the DAE. Therefore, it is desirable to transform the original DAE into an equivalent DAE with lower index [35].

There are several definitions for index concepts. Some of them are mentioned below: Kronecker index, differential index, tractability index, perturbation index, geometric index, and strangeness index. The index can become a local concept with different values in different regions. In fact the index may even be undefined at so called singular points, which typically exhibit impasse phenomenon.

Theorem 1. [44] *If a regular DAE (2.2.2) system with constant coefficients is considered, then two matrices U and V exist such as :*

$$UAV = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

where the matrix N is a nilpotent matrix i.e, there exist a positive integer K with $N^K = 0$ and $N^{K-1} \neq 0$.

Now the following definitions can be given:

Definition 1. (Kronecker Index) [20]: It is said that DAE system (2.2.2) has a Kronecker index $=K$ if A is singular. If A is nonsingular then the Kronecker index $= 0$.

The Kronecker index is also called the Nilpotency index. Hence the ODE has a zero Kronecker index.

Definition 2. (Differential index) [44]: The nonlinear DAEs (2.2.1) has differentiation index m , if m is the minimal number of analytical differentiations:

$F(t, x, \dot{y}) = 0, \frac{dF(t, x, \dot{y})}{dt}, \dots, \frac{d^m F(t, x, \dot{y})}{dt^m} = 0$, such that (2.2.1) allows us to extract an explicit ordinary differential system by algebraic manipulations $\dot{x} = a(x)$ (which is called the underlying ordinary differential equation (*UODE*)).

2.4 Solvability of DAEs

Definition 3. [43] A solution of (2.2.1) is a function $x_* : I \longrightarrow R^n$ if $x_* \in C^m(I, R^n)$ is with

$$F(t, x_*, \dot{x}_*) = 0,$$

where I denotes the open interval, R^n denotes the real vector space and C^m is the space for all m differentiable functions from I into R^n . The formula (2.2.1) is called solvable if it has at least one solution.

Since the dynamics of DAEs are constrained to a set which often is only given implicitly, many theoretical and numerical difficulties arise, which may lead to a sensitive behavior of the solution of DAEs to perturbation in the data. The difficulties are characterized by main ideas for differential algebraic equations such as regularity, index, solution subspace, or hidden constraints, which do not appear for ODEs.

The arbitrarily small perturbations deal with loss of these properties. As a consequence, restrictions to the allowed perturbations usually have to be made, leading to robustness questions for DAEs that are very different from those for ODEs. DAE systems are very complex and difficult to solve, this difficulty which arises the solution is called the solution set.

Additionally, it is said that the DAE has a unique solution through x_* at t_* if any two solutions x_* and $\tilde{x}(t)$ are defined on the open intervals I and $\tilde{I}$, respectively. Both of them comprise t_0 such that $x(t_*) = \tilde{x}_{t_*} = x_*$ verify $x(t) = \tilde{x}(t)$ for all $t \in I \cap \tilde{I}$.

The solution is said to be locally unique if the condition $x(t_*) = \tilde{x}(t_*) = x_*$ implies $x(t) = \tilde{x}(t)$ on some neighborhood of t_* . For autonomous DAEs (that is, problems of the form (2.2.1) with F independent of t , the consistency of an initial value does not

depend on the choice of t_* , and the solution set can be thought of as lying on x -space. In many cases, the solution set has a manifold structure, solutions are uniquely defined through every point on this manifold, and they yield a C^1 -flow on it. This leads to the solution manifold concept and to several definitions in the literature around the notions of a solvable DAE and a regular DAE.

2.5 Reduction of DAE

Reduction is the most suitable way to deal with some formulas of DAE system like quasi linear and Hessenberg [23].

From algebraic equations constraints, DAE can be transformed into ODE by repeating derivations of algebraic equations $g(t, x, \dot{x}) = 0$ with respect to, t so it is easy to apply classical theorems of ODEs.

The index reduction [38] is one type of reduction and it is the best way to reduce the index two which is hessenberg system that it takes the form:

$$\dot{x} = f(x, y), \tag{2.5.1}$$

$$0 = g(x).$$

Now the following examples to illustrate the idea of index reduction can be given:

Example 1. [23] Consider the semi-explicit DAEs system

$$\dot{x} = f(x, y), \tag{2.5.2}$$

$$0 = g(x, y),$$

differentiate the constraint g with respect to time t we get :

$$g_x \dot{x} + g_y \dot{y} = 0.$$

with g_y is nonsingular if g_y^{-1} exist then we will get:

$$\dot{y} = -g_y^{-1} g_x \dot{y}, \quad (2.5.3)$$

So the index of this system is one.

The following example discusses the difference between the semi explicit formula of autonomous equation in (2.2.3) return and the reduced ODE system (2.5.3).

Example 2. *Consider the higher index DAE as the following*

$$\frac{dx}{dt} = x^2 - y$$

$$g(x, y) = 0 = x^3 + x$$

we should differentiate the given semi explicit DAE more than one time to get the reduce of this system,so another constraint will arise, which is

$$3x^2 + 1 = 0.$$

This case is called the hidden constraint.

2.6 Stability of DAEs

Stability concept in DAEs is identical to the stability in ODEs. But in DAEs, the classical stability method of the ODEs theorems can not be applied because of the algebraic

constraint. These theorems can be applied after reduction of system to an ODEs. This reduction is not sometimes easy or may be impossible. In addition, this reduction may change the structure and the state of the stability of the original systems. The Lyapunov stability of a system with respect to its equilibrium of interest is about the behavior of the system outputs toward the equilibrium nearby and around the equilibrium. This is called Stability in the sense of Lyapunov[8].

Definition 4. This equilibrium is said to be Lyapunov stable, if for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$ then for every $t \geq 0$ there is $\|x(t) - x_e\| < \epsilon$.

Definition 5. The equilibrium of the DAE system is said to be asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\lim_{t \rightarrow \infty} \|x(t) - x_e\| = 0$.

Definition 6. The equilibrium of the DAE system is said to be exponentially stable if it is asymptotically stable and there exists $\alpha > 0, \beta > 0, \delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\beta t}$, for all $t \geq 0$.

2.7 A Relation between SP and DAE and ODE

The singularly perturbed ODE, can directly be integrated to obtain time responses to avoid solving the nonlinear algebraic equations. Song.X. shows possible remodel the DAE by a singularly perturbed ODE to avoid solving the nonlinear algebraic equations so that we can directly integrate the ODE without a reduction process and PTE (Perturb and Taylors Expansion) technique is proposed to build the fast dynamics[46]. Thus, by PTE, a DAE is converted to a singularly perturbed ODE [34, 49].

Thus an approximation of the slow dynamics of the original two-time scale system (1.2.1) is taken $\epsilon = 0$ in (1.2.1) under the following conditions:

The fast dynamics is locally uniformly asymptotically stable, i.e, the fast dynamics has all eigenvalues with strictly negative real parts.

It also implies that Jacobian g_y^{FD} is nonsingular and all trajectories remain in the attractive region of the stable equilibrium of the jacobian matrix of fast dynamics.

Under these conditions when $\epsilon = 0$, the dynamics of y is infinitely faster than those of x , the singular perturbation ODE becomes DAE. The slow sub system (*DAE*) approximates the dynamics of the original ODE system

$$\dot{x} = f(x, y, \epsilon),$$

$$\epsilon \dot{y} = g(x, y, \epsilon),$$

well and the dynamic behaviors of the system around an equilibrium point are characterized by the reduced Jacobian matrix Jr of the form

$$Jr = f_x - f_y g^{-1} y g_x.$$

The solution of DAE does not always depend on the boundary or initial values.

A singularly perturbed discrete control system with two small parameters is formulated as an initial value problem. For ordinary differential equations, initial values have to be prescribed for all variables to determine a unique solution. In the case $\epsilon = 0$ not all components appear in dynamic form and some of initial values are precisely determined by constraints. Thus, no initial values can be prescribed for them. It is always considered to deal with smooth inner functions to obtain outer functions in the solution. With the constraint, the outer ones can not be derivative. In the case of differential equations, boundary conditions cannot be satisfied, in algebraic equations, the possible number of solutions is decreased [46].

2.8 Bifurcation

Bifurcation analysis is widely used in nonlinear system dynamic studies[7]. In general, bifurcations may occur at any point along the parameterized path.

Definition 7. [50, 51]:A bifurcation is a qualitative or topological change in the behavior solution of a dynamical system by a small perturbation of p at a bifurcation point.

Definition 8. [46]:Bifurcation points are critical for dynamic stability analysis of nonlinear systems, which deal with local properties such as the dynamic stability of equilibrium points under small variations of parameter p .

- Stationary bifurcations involve changes in the mechanism of equilibrium point when created and destroyed as parameter varies.
- Periodic bifurcations involve changes in the periodic solutions or limit cycle.

As an it is illustrated in[52], the differential equation is considered as:

$$\dot{x} = kx,$$

in which $k \in R$ is a parameter. The solution can easily given as:

$$x(t) = \exp(kt)x(0).$$

It obviously depends on k , but as the value of k is changed from $k = 1$ to $k = 1.01$, there is no qualitative change in the solution. When, the value of a round zero changes, qualitative change in the behavior can be observed. For $k = 0$ all solutions are constant functions, for negative values of k the solutions tend to zero, while for positive values of k they tend to infinity. All of these bifurcations can occur in continuous dynamical systems of two or higher dimensions.

2.8.1 Equilibrium point

For the following nonlinear planar system,

$$\dot{x} = F(x, p) \tag{2.8.1}$$

where $F : R^2 \longrightarrow R^2$, $(x, p) \in R^2 \times R$. A point x_* is called an equilibrium point of (2.8.1) if $f(x_*) = 0$. If $x(0) = x_*$ then $x(t) = x_*$ for all times.

Some information about this subject is given[5, 14, 36, 40].

- The equilibrium point (x_*, y_*) is a node if the two eigenvalues $\lambda_{1,2}$ of the jacobian matrix ($J = \det F_x$) are real and they have the same sign.
- The equilibrium point (x_*, y_*) is stable if the eigenvalues are negative, otherwise it is unstable

- The equilibrium point (x_*, y_*) is a saddle point if the eigenvalues are real with opposite signs.
- The equilibrium point (x_*, y_*) is a focus (spiral) if the eigenvalues form a complex conjugate pair.
- The equilibrium (x_*, y_*) is said to be hyperbolic if all eigenvalues of the Jacobian matrix have non zero real parts.
- The equilibrium point is said to be non-hyperbolic if at least one eigenvalue of the Jacobian matrix is zero or has a zero real part, and the non-hyperbolic equilibrium when the parameter is $p = p_0$. Then there is a bifurcation at this value of the parameter.

2.8.2 Kinds of Bifurcation

Generally, the local stability properties of equilibria, periodic orbits are at a bifurcation. Bifurcation is two types :

- (1) **Local bifurcations** [41]: They can be analyzed through changes in the local stability properties of equilibria or periodic orbits as parameters cross through thresholds.

Definition for some kinds of Bifurcation are given . The First three kinds depend on the branch of equilibrium point.

Definition 9. Fold (Saddle node bifurcation)[52]: Unique curve equilibrium points passed through the origin point. Moreover, the curve lies entirely on one side of $p = 0$, in the $p - x$ plane.

Definition 10. Transcritical bifurcation[52]: Two curves of critical points intersect at the origin in the $p - x$ plane. Both curves exist on either side of $p = 0$.

Definition 11. Pitchfork bifurcation[52]: Two curves of equilibrium points intersect at the origin in the $p - x$ plane. Only one curve $x = 0$ exist on both sides of $p = 0$.

The following definition of the two bifurcations depend on eigenvalues.

Definition 12. Singularity Induced bifurcation[46]: If the jacobian matrix at the origin J^0 has an eigenvalue increasing to infinity, the origin is called singularity induced bifurcation(SIB).

Definition 13. Hopf bifurcation(HP)[46]: If the jacobian matrix at the origin ($\det(DF(0) = J^0)$) has a simple pair of pure imaginary eigenvalues and there are no other eigenvalues with zero real part then the origin is a Hopf bifurcation point.

(2) Global bifurcations [36]: They often occur when larger invariant sets of the systems collide with each other, or with equilibria of the systems. There is some kinds of global bifurcation homoclinic, hetroclinic, infinite periodic bifurcation, e,t,c. The next figure(2.1)(2.2)(2.3) to explains some kinds of bifurcation[23]:

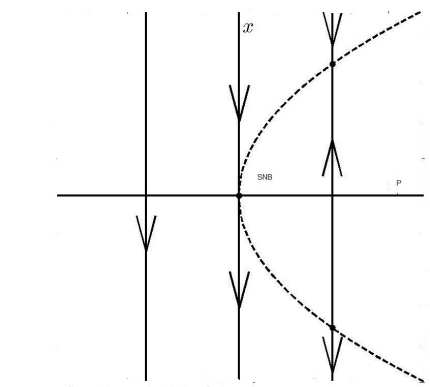


Figure 2.1: Saddle Node Bifurcation

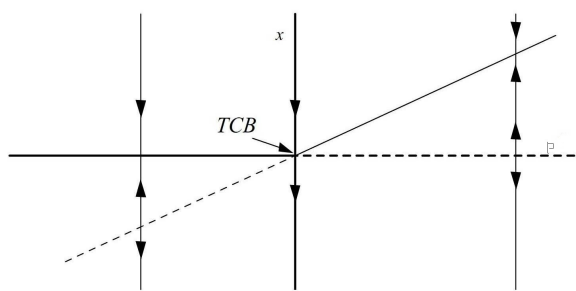


Figure 2.2: Transcritical Bifurcation

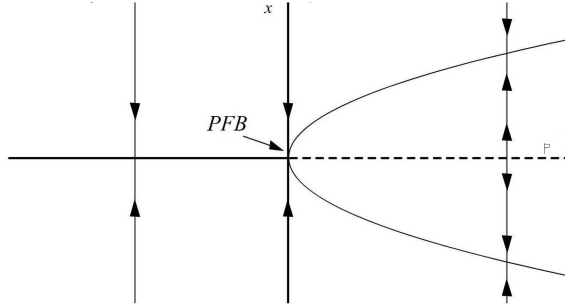


Figure 2.3: Pitchfork Bifurcation

2.9 Blow up method

Initially, the blowup method is a technique which is separately developed from the geometric study of multiple-time-scale equations.

In mathematics, blowing up or blowup is a type of geometric transformation which replaces a subspace of a given space with all the directions pointing out that subspace. For example, the blowup of a point in a plane replaces the point with the projectivized tangent space at that point. The metaphor is that zooming in on a photograph to enlarge a part of the picture, rather than referring to an explosion.

Explicitly, the main idea is that blow up through the change of the variables is not a diffeomorphism, to a higher dimensional structure such as a line, a circle or sphere.

Then for studying the original non degenerate point one studies the new different non-degenerate singularities will appear in a higher dimensional structure. In case that some fixed points of the blown-up system is still non-hyperbolic, the procedure will be repeated. Dumortier proved that this iterative process of desingularization is finite.

A blowup can also be called monoidal transformation, locally quadratic transformation, dilatation, s-process, or Hopf map [3, 29].

2.9.1 Polar Coordinate

We can use polar coordinate to get completely visualized about the blow up method where it is always singular at the origin to obtain the information of the behavior of this solution only near an equilibrium point of reduced system. The polar coordinate is not particularly useful to determine the phase portrait near the origin because the desingularized the vector field in polar coordinates is a smooth extension to the singular line represented as the equator of the phase cylinder. The following definition to explain what is polar coordinates is:

Definition 14. [27] The polar coordinate system is a two dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point that is called the pole (or origin) and an angle from a fixed direction. The point is represented by the ordered pair (r, ϕ) and r, ϕ which are called polar coordinates.

Relationship between derivatives in Cartesian and polar coordinates

Using $x = r\cos\phi$ and $y = r\sin\phi$, one can derive a relationship between derivatives in Cartesian and polar coordinates. For a given function, $u(x, y)$, it follows that :

$$\begin{aligned} r \frac{\partial}{\partial r} &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \phi} &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \end{aligned}$$

Using the inverse coordinates transformation, an analogous reciprocal relationship can be derived between the derivatives . Given a function $u(r, \phi)$, We have the following formulae:

$$\frac{\partial}{\partial x} = \sec \phi \frac{\partial}{\partial r} - \frac{1}{r} \csc \phi \frac{\partial}{\partial \phi}.$$

$$\frac{\partial}{\partial y} = \csc \phi \frac{\partial}{\partial r} + \frac{1}{r} \sec \phi \frac{\partial}{\partial \phi}.$$

2.10 The implicit function theorem

The implicit function theorem in finite dimensions is concerned with a system of equations of the form [6]:

$$f_i(x_1, \dots, x_n, p_1, \dots, p_k) = 0, i = 1, \dots, n : \quad (2.10.1)$$

Equation (2.10.1) can be written in vector form:

$$F(x, p) = 0, \quad (2.10.2)$$

where $F : R^n \times R^k \longrightarrow R^n$. The Jacobian matrix for (2.10.2) is

$$J(x, p) = \left(\frac{\partial f_i}{\partial x_j}(x, p) \right)_{i,j=1,\dots,n} \quad (2.10.3)$$

We shall work in a neighborhood of a critical point $(x_0, p_0) \in R^n \times R^k$.

Theorem 2. *Implicit Function Theorem [6]:* Let F be given in (2.10.2) above, suppose that

$$F(x_0, p_0) = 0$$

and

$$\det J(x_0, p_0) \neq 0 :$$

Then there exists a neighborhoods U of x_0 in R^n and V of p_0 in R^k and a function $X : V \longrightarrow U$ such that for every $p \in V$, (2.10.2) has the unique solution $x = X(p)$ in U . Moreover if $F \in C^s(R^n)$ then $X \in C^s(R^k)$. In symbols

$$F(X(p), p) = 0,$$

$$X(p_0) = x_0.$$

Chapter 3

Bifurcation Theory in Singularity Perturbed ODEs by Blow up Method

3.1 Introduction

Fast-Slow system is ordinary differential equations (*ODEs*) in which some variables have derivatives of much larger magnitude than those of other variables. Fast-slow systems are characterized by two different time scales, fast and slow time. The easiest case is a separation, which splits the variables into two groups[19]. First the standard form is considered:

$$\begin{aligned}\frac{dx}{dt} &= \dot{x} = f(x, y, \epsilon), \\ \frac{dy}{dt} &= \dot{y} = \epsilon g(x, y, \epsilon),\end{aligned}\tag{3.1.1}$$

where $f : R^n \times R^m \times R \longrightarrow R^n$, $g : R^n \times R^m \times R \longrightarrow R^m$ and typically $0 < \epsilon \ll 1$. Systems (3.1.1) are equivalent with the time rescale $t = \tau/\epsilon$. The form of singularity in (3.1.1) suggests the rescaling $\tau = \epsilon t$. This transform (3.1.1) to

$$\begin{aligned}\frac{dx}{d\tau} &= \epsilon f(x, y, \epsilon), \\ \frac{dy}{d\tau} &= g(x, y, \epsilon),\end{aligned}\tag{3.1.2}$$

where t denotes the slow time and τ represents the fast time. Along with (3.1.1) and (3.1.2), it is convenient to consider the equivalent systems in which ϵ is introduced as a dummy variable ,i.e.,the value of ϵ is 0 or 1 in the phase space . The systems are

$$\epsilon \dot{x} = f(x, y, \epsilon),\tag{3.1.3}$$

$$\dot{y} = \epsilon g(x, y, \epsilon),$$

$$\dot{\epsilon} = 0,$$

and

$$\dot{x} = f(x, y, \epsilon),\tag{3.1.4}$$

$$\dot{y} = \epsilon g(x, y, \epsilon),$$

$$\dot{\epsilon} = 0.$$

The suitable way to analyze a fast slow system is to consider the case $\epsilon = 0$, we get two formulations from (3.1.1) and (3.1.2) as the follows:

Definition 15. [29] The parameterized system of singular ODEs which is obtained by taking the limit case $\epsilon = 0$ on the fast time scale formulation (3.1.5), is called a fast subsystem or fast vector field:

$$\dot{x} = f(x, y, 0),\tag{3.1.5}$$

$$\dot{y} = 0.$$

Definition 16. [29] The equation (3.1.2) becomes at the limit case $\epsilon = 0$ the reduced problem (3.1.6) which is the set of differential algebraic equation:

$$0 = f(x, y, 0), \tag{3.1.6}$$

$$\dot{y} = g(x, y, 0),$$

the above formulation is called slow subsystem where f and g are sufficiently smooth $x \in R^n, y \in R^m$ and $\epsilon \in R$.

Definition 17. [3] The set C is called critical. It is the set of critical point of the layer problem where the limit case $\epsilon = 0$ is taken. The critical set is referred to by C where

$$C = \{(x, y) \in R^2, f(x, y, 0) = 0\}$$

if $C \subset R^n \times R^m$ then it is called smooth critical manifold.

Consider the parameterized singular perturbation system (3.1.1) given by:

$$\dot{x} = f(x, y, \epsilon, p), \tag{3.1.7}$$

$$\epsilon \dot{y} = g(x, y, \epsilon, p),$$

where $f, g : R^2 \times R \times R \rightarrow R^2$, p is the parameter in R . We assume here there is always a neighborhood for the bifurcation point where both f and g are continuous in their common points at the neighbor. Let C be the set of critical points of (3.1.7) where

$$S = \{(x, y, p, \epsilon) \in R^2 \times R \times R, f(x, y, p, \epsilon) = g(x, y, p, \epsilon) = 0\}. \tag{3.1.8}$$

And a function F is defined by:

$$F(x, y, p, \epsilon) = \begin{bmatrix} f(x, y, p, \epsilon) \\ g(x, y, p, \epsilon) \end{bmatrix}, \quad (3.1.9)$$

then the solution set for $F(x, y, p, \epsilon) = 0$ is of equilibrium point. To arrange our work we will consider that the origin point $(0, 0, 0, 0)$ is the bifurcation point of the system for the singularity perturbation system (*i.e*) $f_0 = 0$ and $g_0 = 0$. To find a bifurcation at another point which is not $(0, 0, 0, 0)$, we will make shifting for the coordinates.

Bifurcation point should be non hyperbolic. Thus the Jacobian matrix $f_x(x, y, p, \epsilon)$ of the non hyperbolic singular perturbation system should be singular, i.e $Det(f_x = 0)$, that means the Jacobian will have a row or column with zero values. A geometric technique to desingularize non-hyperbolic points is the blow-up method.

3.2 Polar Blow Method in Singular perturbation System

In [29], the polar blow up method is given on R^2 . In this chapter, polar blow up coordinate transformation on R^3 is given. The following definition of polar blow up for singular perturbation ODEs:

Definition 18. Polar blow up is denoted by ψ from the manifold $A = S^2 \times I$ into R^3 where

$$\psi(r, \phi, \epsilon) = (r \cos \phi, r \sin \phi, \epsilon),$$

where I is an interval in R with $0 \in I$, and $\psi \in [0, 2\pi)$ parameterizes S^2 . The polar blow

up of the vector field F :

$$F(x, y, \epsilon) = \begin{bmatrix} f(x, y, \epsilon) \\ g(x, y, \epsilon) \end{bmatrix}, \quad (3.2.1)$$

is denoted by $\hat{F}$ where:

$$\hat{F} = ((D\psi)^{-1}F(\psi(r, \phi, \epsilon))), \quad (3.2.2)$$

and the polar blow up of f and g is:

$$\hat{f}(r, y\epsilon) = ((D\psi)^{-1} \circ f \circ \psi)(r, \phi, \epsilon), \quad (3.2.3)$$

$$\epsilon\hat{g}(r, \phi, \epsilon) = ((D\psi)^{-1} \circ g \circ \psi)(r, \phi, \epsilon), \quad (3.2.4)$$

where

$$(D\psi)^{-1} = \begin{bmatrix} -\frac{\sin\phi}{r} & \frac{\cos\phi}{r} & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for $r \neq 0$ and by continuous extension of $\hat{F}$ to $r = 0$. F is a vector field on $S^2 \times I$.

If Eq (3.1.1) a vector field is with $J_l(F) = 0$ for $l \in \{0, 1, \dots, k\}$ and $J_{k+1}(\hat{F}) \neq 0$, where $J_l(F)$ is the Jacobian matrix for F and $J_{k+1}(\hat{F})$ is Jacobian matrix for $\hat{F}$. The (rescaled) polar blowup is defined:

$$\bar{F} = \frac{1}{r}\hat{F}, \quad (3.2.5)$$

and

$$\begin{aligned} \bar{f}(r, \phi, \epsilon) &= \frac{1}{r}\hat{f}(r, \phi, \epsilon) \\ \bar{g}(r, \phi, \epsilon) &= \frac{1}{r}\hat{g}(r, \phi, \epsilon). \end{aligned}$$

Remark 1. If the map ψ on manifold is differentiable then the push forward

$$\psi_* : T_A \longrightarrow R^3$$

induces a vector field $\hat{F}$ on $A = S^2 \times I$ if we require the condition

$$\psi_*(\hat{F}) = F,$$

where T_A is tangent bundle, for each smooth function ψ is defined near the point b .

The term rescaled is introduced to emphasize that scaling by a factor $1/r^k$ does not change the qualitative structure of the vector field near the singularity at $r = 0$. However, the times on orbits of $\hat{F}$ and $\bar{F}$ are not comparable. The term rescaled is dropped if it is clear that we want to desingularize the vector field under consideration[29].

The construction of polar blow up can be generalized and S^{n-1} is denoted by the $(n-1)$ dimensional unit sphere in R^n instead of choosing an explicit polar parameterizations. Now the following theorem which is proved in [29] for the ODEs but here we proved it on singular perturbation theory.

Theorem 3. *Let F be a C^∞ vector field on R^n with $F(0) = 0$. Then the polar blow up is a C^∞ vector field on $S^{n-1} \times I$.*

Proof. Define the vector field on R^n

$$\begin{aligned}\dot{r} &= \sum_{i=1}^n z_i \dot{z}_i + \dot{\epsilon} \\ \dot{\phi} &= \frac{1}{2}(z_i \dot{z}_j - z_j \dot{z}_i) \\ \dot{\epsilon} &= 1,\end{aligned}$$

where $z_i = \{x_1, x_2, \dots, x_n\}$, $z_j = \{y_1, y_2, \dots, y_m\}$. Using these facts, we will show that:

$$r^2 F = \langle \dot{r}, F \rangle \dot{r} + 2 \sum \langle \dot{\phi}, F \rangle \dot{\phi} + \sum \langle \dot{\epsilon}, F \rangle \quad (3.2.6)$$

on $S^{n-1} \times I$.

Here $\dot{r}$ represents the radial component and $\dot{\phi}$ the rotational components of the vector field F on $S^{n-1} \times I$. Define $\hat{r}$, $\hat{\phi}$ and $\hat{\epsilon}$ by:

$$\psi_*(\hat{r}) = \dot{r},$$

$$\psi_*(\hat{\phi}) = \dot{\phi},$$

$$\psi_*(\hat{\epsilon}) = \dot{\epsilon}.$$

Auxiliary functions are defined as:

$$\alpha_1(r, \phi, \epsilon) = \langle \dot{r}, F \rangle = \frac{1}{r^2} (\cos \phi f(r \cos \phi, r \sin \phi, \epsilon) + g \sin \phi \sin \Theta(r \cos \phi, r \sin \phi, \epsilon),$$

$$\alpha_2(r, \phi, \epsilon) = \langle \dot{\phi}, F \rangle = \frac{1}{r^2} (-r \sin \phi f(r \cos \phi, r \sin \phi, \epsilon) + r \cos \phi g(r \cos \phi, r \sin \phi, \epsilon)), \text{ and}$$

$$\alpha_3(r, \phi, \epsilon) = \langle \dot{\epsilon}, F \rangle = 0$$

on taking the push forward in equation (3.2.6) we get:

$$r^2 F = \psi_*(\alpha_1 \circ \psi) \hat{r} + 2 \sum (\alpha_2 \circ \psi) \hat{\phi} + \sum (\alpha_3 \circ \psi) \hat{\epsilon}$$

$$F = \psi_* \left(\frac{1}{r^2} (\alpha_1 \circ \psi) \hat{r} + 2 \sum (\alpha_2 \circ \psi) \hat{\phi} + \sum (\alpha_3 \circ \psi) \hat{\epsilon} \right)$$

$$D\psi^{-1} \circ F \circ \psi = \frac{1}{r^2} ((\alpha_1 \circ \psi) \hat{r} + 2 \sum (\alpha_2 \circ \psi) \hat{\phi} + \sum (\alpha_3 \circ \psi) \hat{\epsilon}),$$

this equation is defined for $r = 0$ at $0 \in R^n$. It is known that $F, \dot{r}, \dot{\phi}$ and $\dot{\epsilon}$ vanish. When the product rule is used α_1, α_2 and α_3 have non linear term in Taylor expansion, $\alpha_1 \circ \psi$,

$\alpha_2 \circ \psi$, and $\alpha_3 \circ \psi$ $\frac{\partial(\alpha_1 \circ \psi)}{\partial r}$, $\frac{\partial(\alpha_2 \circ \psi)}{\partial r}$ and $\frac{\partial(\alpha_3 \circ \psi)}{\partial r}$ are zero. This implies that $(1/r^2)(\alpha_1 \circ \psi)$, $(1/r^2)(\alpha_2 \circ \psi)$ and $(1/r^2)(\alpha_3 \circ \psi)$ are C^∞ for $r = 0$. Hence, the vector field (3.1.1) is a well-defined vector field for $r = 0$. $\square$

The polar blow up will be applied on following example to transform the degenerate point to higher dimensional case as a sphere.

Example 3. Consider the singular parameterizations system

$$\dot{x} = x^2$$

$$\dot{y} = \epsilon(y^2 - xy)$$

since $F(0, 0, 0) = 0$, and $DF(0, 0, 0) = 0$ then $(0, 0, 0)$ is non hyperbolic equilibrium point. We will destructuralize this point by polar blow up

$$\hat{F} = (D\psi)^{-1}(F \circ \psi), \text{ where} \quad (3.2.7)$$

$$(D\psi)^{-1} = \begin{bmatrix} \frac{-\sin\phi}{r} & \frac{\cos\phi}{r} & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and} \quad (3.2.8)$$

$$F \circ \psi = \begin{bmatrix} r^2 \cos^2 \phi \\ \epsilon(r^2 \sin^2 \phi - r^2 \cos \phi \sin \phi) \\ 0 \end{bmatrix}, \text{ then,} \quad (3.2.9)$$

$$\begin{bmatrix} \hat{f}(r, \phi, \epsilon) \\ \hat{g}(r, \phi, \epsilon) \\ \hat{e}(r, \phi, \epsilon) \end{bmatrix} = \begin{bmatrix} -r \sin \phi \cos^2 \phi + r \epsilon (\sin^2 \phi \cos \phi - \cos^2 \phi \sin \phi) \\ r^2 \cos^3 \phi + r^2 \epsilon (\sin^3 \phi - \cos \phi \sin^2 \phi) \\ 0 \end{bmatrix}.$$

Since $J_1(F) = 0$ and $J_2(\hat{F}) \neq 0$ then we will divide $\hat{F}$ by r to obtain $\bar{F}$ as the following:

$$\bar{F} = \frac{\hat{F}}{r} = \begin{bmatrix} -\sin\phi\cos^2\phi + \epsilon(\sin^2\phi\cos\phi - \cos^2\phi\sin\phi) \\ r\cos^3\phi + \epsilon r(\sin^3\phi - \cos\phi\sin^2\phi) \\ 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\phi} \\ \dot{\epsilon} \end{bmatrix}.$$

If we put $\bar{F} = 0$ we get $r = 0$ and $\phi = 0, \mp\frac{\Pi}{2}, \mp\Pi$

$$D\bar{F}(0,0,0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $(0,0,0)$ is saddle. This idea explained in figures(3.1)(3.2)

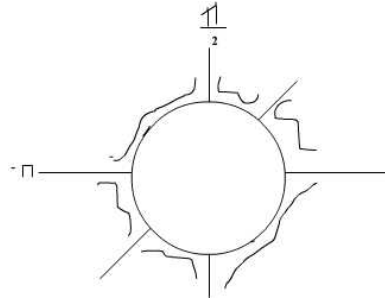


Figure 3.1: a:Phase portrait blown-up vector field $\bar{F}$

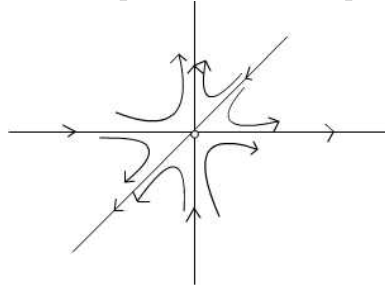


Figure 3.2: b:Phase portrait for original vector field F

3.3 Fold (Saddle node)Bifurcation in Singular Perturbation System

These bifurcations give rise to trajectories that make jump transitions from an equilibrium manifold of an ϵ approach to zero. A fold bifurcation point is a point where a pair of equilibria meets and disappears with a zero eigenvalue. One of the equilibria (node) is stable while the other (saddle) is unstable.

In a Fold bifurcation, the region of attraction of the stable equilibrium point shrinks

due to an approaching the unstable equilibrium point and the stability is eventually lost when the two equilibria coalesce and disappear. Now consider the singularly ODEs (3.1.7) and the equation (3.2.5) which we get it by blowing up ODEs (3.1.7) where

$$F(x, y, p, \epsilon) = \begin{bmatrix} f(x, y, p, \epsilon) \\ g(x, y, p, \epsilon) \end{bmatrix}$$

and $\hat{F}$ as given in (3.2.2). By Blow up method, the curve of equilibrium points should satisfy the following two conditions to have Fold bifurcation around ϕ axis at the origin point:

1. It should be to the tangent of line $p = 0$ at $\phi = 0$, i.e, $p_\phi^0 = 0$,
2. It lies entirely on one side of $p = 0$, i.e:

$$p_{\phi\phi}^0 \neq 0.$$

Theorem 4. *Consider the singularity perturbation system (3.1.7) with the set of equilibrium points S . If the following conditions hold:*

1. $\bar{f}_p^0 \neq 0$, and
2. $\bar{f}_{\phi\phi}^0 \neq 0$,

where $\bar{f}$ as given in (3.2.3) then $(0, 0, 0, 0)$ is Fold (saddle node) bifurcation point.

Proof. By condition 1 above

$$\bar{F}_p^0 \neq 0,$$

then by implicit function theorem we have:

$$\bar{F}(r, \phi, p(r, \phi, \epsilon), \epsilon) = 0. \tag{3.3.1}$$

Eq(3.3.1) w.r.t ϕ is Differentiated, we get :

$$\bar{F}_\phi + \bar{F}_r \dot{r} + \bar{F}_p(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) + \bar{F}_\epsilon(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) = 0.$$

When there is evaluation at the origin, we obtain:

$$\bar{F}_\phi^0 + \bar{F}_p^0 p_\phi^0 + \bar{F}_\epsilon^0 \epsilon_\phi^0 = 0.$$

Now by applying condition 1 above and assumption $\bar{f}_\phi^0 = 0$, we obtain:

$$p_\phi^0 = -(\bar{f}_\phi^0 + \bar{f}_\epsilon^0 \epsilon_\phi^0)(\bar{f}_p^0)^{-1} = 0$$

Now take the second derivative w.r.t ϕ :

$$\begin{aligned} & \bar{F}_{\phi\phi} + \bar{F}_{\phi r} \dot{r} + \bar{F}_{\phi p}(p_\phi + p_\epsilon \dot{\epsilon} + p_r \dot{r}) + \bar{F}_{\phi\epsilon}(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) + \bar{F}_r \dot{r} + [\bar{F}_{r\phi} + \bar{F}_{rr} \dot{r} + \\ & \bar{F}_{rp}(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) + \bar{F}_{r\epsilon}(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p})] \dot{r} + \bar{F}_p[p_{\phi\phi} + p_{\phi r} \dot{r} + p_{\phi\epsilon}(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) + p_r \dot{r} + \\ & (p_{r\phi} + p_{r\epsilon}(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) + p_{rr}) \dot{r}] + [\bar{F}_{p\phi} + \bar{F}_{pr} \dot{r} + [p_\phi + p_\epsilon(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) + p_r \dot{r}] \bar{F}_{p\epsilon}(\epsilon_\phi + \epsilon_r \dot{r} + \epsilon_p \dot{p}) + \\ & F_{pp}(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon})] + \bar{F}_\epsilon[\epsilon_{\phi\phi} + \epsilon_{\phi r} \dot{r} + \epsilon_{\phi p}(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) p_r \dot{r} + (\epsilon_{r\phi} + \epsilon_{rp}(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) + \epsilon_{rr}) \dot{r}] \\ & + [\bar{F}_{\epsilon\phi} + F_{\epsilon r} \dot{r} + [\epsilon_\phi + p(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) + p_r \dot{r}] \bar{F}_{\epsilon p}(p_\phi + p_r \dot{r} + p_\epsilon \dot{\epsilon}) + F_{\epsilon\epsilon}(\epsilon_\phi + p_r \dot{r} + p_p \dot{p})] = 0. \end{aligned}$$

Evaluating at $(0, 0, 0, 0)$ and substituting $p_\phi^0 = 0$ as we proved, implies:

$$\begin{aligned} & \bar{F}_{\phi\phi}^0 + F_r^0 \dot{r} + \bar{F}_p^0 p_{pp}^0 + F_p^0 p_r^0 \dot{r} + \bar{F}_\epsilon^0 \epsilon_{\phi\phi} \bar{F}_\epsilon^0 + \bar{F}_\epsilon^0 \epsilon_r^0 \dot{r} = 0, \\ & f_{\phi\phi}^0 + (f_r^0 + f_p^0 p_r^0) \dot{r} + f_p^0 p_{\phi\phi}^0 + (\bar{f}_r^0 + \bar{f}_\epsilon^0 \epsilon_r^0 + \bar{f}_\epsilon^0 \epsilon_{\phi\phi}^0) = 0. \end{aligned}$$

Then depending on condition 1, it can be written:

$$p_{\phi\phi}^0 = -[f_{\phi\phi}^0 + (f_r^0 + f_p^0 p_r^0) \dot{r} + ((f_r^0 + f_\epsilon^0 \epsilon_r^0) \dot{r})](f_p^0)^{-1}.$$

Last, with both conditions 1 and 2, it is certainly $p_{\phi\phi}^0 \neq 0$, which ends the proof. $\square$

Example 4. Consider the fast-slow system

$$\dot{x} = \epsilon(y^2 - xy), \quad (3.3.2)$$

$$\dot{y} = -px^2 + y^2.$$

where $\dot{x} = \frac{dx}{d\tau}$ and $\dot{y} = \frac{dy}{d\tau}$, when we apply polar blow up as the following:

$$\bar{F} = \frac{1}{r}\hat{F}$$

where $\hat{F} = (D\psi)^{-1}(F \circ \psi)_{(r,\phi,\epsilon)}$, we obtain

$$\dot{r} = -\epsilon \sin^3 \phi + \epsilon \cos \phi \sin^2 \phi - p \cos^3 \phi + \sin^2 \phi \cos \phi,$$

$$\dot{\phi} = \epsilon r \cos \phi \sin^2 \phi - \epsilon r \cos^2 \phi \sin \phi - p r \sin \phi \cos^2 \phi + r \sin^3 \phi,$$

$$\dot{\epsilon} = 0.$$

Since $\bar{f}_p^0 \neq 0$, $\bar{f}_{\phi\phi}^0 \neq 0$, and $\bar{g}_r^0 = 0$, $\bar{f}_r^0 = 0$, $p_r^0 = 0$, they cancel the second term in condition 2 and leave it is value not equal to zero as required:

$$\hat{f}(r, \phi, \epsilon) = 0, \text{ and}$$

$$\hat{g}(r, \phi, \epsilon) = 0.$$

Hence, if $p = 0$ we get two equilibrium points: $(0, 0, 0, 0)$, $(\Pi, 0, 0, 0)$. Finally if $p > 0$ then there is one equilibrium point $(0, \frac{\Pi}{2}, \epsilon, p)$, but if $p < 0$ becomes $(0, \frac{\Pi}{2}, \epsilon, -p)$. We take, s the following figures(3.3)(3.4) to understand this example.

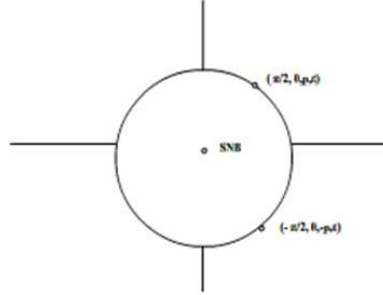


Figure 3.3: Fold bifurcation for $\bar{F}$

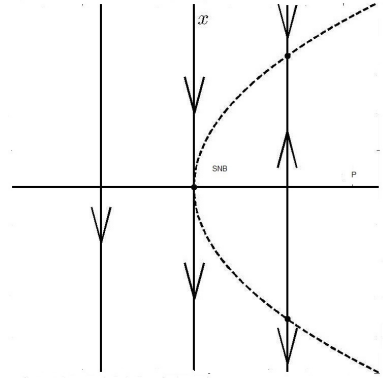


Figure 3.4: Fold bifurcation for F

3.4 Transcritical Bifurcation in Singular Perturbation System(TCB)

As it is known in Transcritical bifurcation (TCB), there should be two curves passing through the origin and that $F_p = 0$. So we can not use the implicit function theorem, hence we write the function F given in (3.1.7) as bellow where the functions M and N

are defined as following [26]:

$$f(x, y, p, \epsilon) = xM(x, y, p, \epsilon),$$

$$g(x, y, p, \epsilon) = xN(x, y, p, \epsilon),$$

where $M : R^n \times R^m \times R \times R \longrightarrow R^n$, $N : R^n \times R^m \times R \times R \longrightarrow R^m$ and p is the bifurcation parameter, $x \in R^m$ and $y \in R^n$, $0 < \epsilon \ll 1$.

Since

$$F(x, y, p, \epsilon) = \begin{bmatrix} f(x, y, p, \epsilon) \\ g(x, y, p, \epsilon) \end{bmatrix},$$

then

$$F(x, y, p, 0) = \begin{bmatrix} xM(x, y, p, 0) \\ xN(x, y, p, 0) \end{bmatrix}$$

and $M(0, y, p, \epsilon) = \frac{\partial f}{\partial x}$ at $x = 0$, $N(0, y, p, \epsilon) = \frac{\partial g}{\partial x}$ at $x = 0$.

So $x = 0$ is a surface of critical points passing through the origin from side to side of $p = 0$. Hence, all what we need to do is finding another surface, in $M = 0$ or $N = 0$, passing through the origin from side to side of $p = 0$, (we need to prove $p_x^0 = 0$.)

In this section, we consider $x = r\cos\phi$, $y = r\sin\phi$, $\epsilon = \epsilon$ and the non hyperbolic point is $\phi = 0$, $r = 0$, $p = 0$, $\epsilon = 0$.

Theorem 5. *Consider the singularity perturbation ODEs (3.1.7) with the surface of equilibrium point S . If*

1. $\bar{f}_p^0 = 0$,
2. $\bar{f}_{\phi p}^0 \neq 0$, and

$$3. \bar{f}_{\phi\phi}^0 \neq 0,$$

then $(0, 0, 0, 0)$ is a Transcritical bifurcation point.

Proof. Consider the function $M(r, \phi, p, \epsilon) = f_\phi$. Since $\bar{f}_{\phi p}^0 \neq 0$ then by defining the function M :

$$M_p(r, \phi, p, \epsilon) \neq 0,$$

and by implicit function theorem:

$$M(r, \phi, p(r, \phi, \epsilon), \epsilon) = 0,$$

Differentiating w.r.t ϕ , we get:

$$M_\phi(r, \phi, p, \epsilon) + M_r(r, \phi, p, \epsilon)\dot{r} + M_p(r, \phi, p, \epsilon)(p_\phi + p_r(r, \phi, \epsilon)\dot{r} + p_\epsilon(r, \phi, \epsilon)(\epsilon_\phi + \epsilon_r\dot{r} + \epsilon_p\dot{p})) = 0.$$

If we evaluate at the origin, and because of $\bar{f}_\phi^0 = 0$, we get:

$$\bar{f}_{\phi\phi}^0(r, \phi, p, \epsilon) + \bar{f}_{\phi p}^0(r, \phi, p, \epsilon)p_\phi^0(r, \phi, \epsilon) + \bar{f}_{\phi\epsilon}^0(r, \phi, p, \epsilon)\epsilon_\phi^0(r, \phi, p) = 0.$$

Then by conditions 2 and 3, we get:

$$p_\phi^0(r, \phi, \epsilon) \neq 0$$

which ends the proof. □

3.5 Pitchfork Bifurcation in Singular Perturbation System

To get a pitchfork bifurcation at the origin we need to find two surfaces of critical points passing through the origin. One is from side to side of $p = 0$ and the other should be put

entirely in one side of $p = 0$, i.e. $p_\phi^0 = 0$ and $p_{\phi\phi}^0 \neq 0$. The following theorem formulate the conditions of Pitchfork bifurcation to be occurred.

Theorem 6. *Consider the singular perturbation system (3.1.7) with the surface of equilibrium point S . If*

1. $\bar{f}_\phi^0 \neq 0$,
2. $\bar{f}_{\phi\phi}^0 = 0$,
3. $\bar{f}_{\phi p}^0 \neq 0$,
4. $\bar{f}_{\phi\phi\phi}^0 \neq 0$,

then $(0, 0, 0, 0)$ is a pitchfork bifurcation point.

Proof. Consider the function $M(r, \phi, p, \epsilon) = f_\phi$ which is given in section (3.4). Since $f_{\phi p}^0 \neq 0$ so by definition of the function M :

$$M_p(r, \phi, p, \epsilon) \neq 0.$$

Then by implicit function theorem:

$$M(r, \phi, p(r, \phi), \epsilon) = 0, \tag{3.5.1}$$

which can be differentiated w.r.t ϕ .

$$M_\phi(r, \phi, p(r, \phi), \epsilon) + M_r(r, \phi, p(r, \phi), \epsilon)\dot{r} + M_p(r, \phi, p(r, \phi), \epsilon)(p_\phi(r, \phi, \epsilon) + p_r(r, \phi, \epsilon)\dot{r}) +$$

$$M_\epsilon(\epsilon_\phi + \epsilon_r\dot{r} + \epsilon_p\dot{p}) = 0.$$

If we evaluate at the origin, we get:

$$\bar{f}_{\phi\phi}^0(r, \phi, p, \epsilon) + \bar{f}_{\phi p}^0(r, \phi, p, \epsilon)p_\phi^0(r, \phi, \epsilon, p) + \bar{f}_{\phi\epsilon}^0\epsilon_\phi^0(r, \phi, p, \epsilon) = 0.$$

By conditions 2 and 3, we get

$$p_\phi^0(r, \phi, \epsilon) = 0.$$

it is kept to prove $p_{\phi\phi}^0 \neq 0$. To do so we differentiate (3.5.1) two times w.r.t ϕ :

$$\begin{aligned} M_{\phi\phi} + M_{\phi\phi r}\dot{r} + M_{\phi p}(p_\phi + p_r\dot{\phi} + p_\epsilon\dot{\epsilon}) + M_{\phi\epsilon}(\epsilon_\phi + \epsilon_r\dot{r} + \epsilon_p\dot{p} + p_\epsilon) + M_r\dot{r} + [M_{r\phi} + M_{rr}\dot{r} + M_{rp}(p_\phi + p_r\dot{r} + p_\epsilon\dot{\epsilon} + \\ M_\epsilon(\epsilon_\phi + \epsilon_r\dot{r} + \epsilon_p\dot{p}))]\dot{r} + M_p[p_{\phi\phi} + p_{\phi r}\dot{r} + p_r\dot{r} + (p_{r\phi} + p_{rr}\dot{r})\dot{r}] + [M_{p\phi} + M_{pr}\dot{r} + M_{pp}(p_\phi + p_r\dot{r})][p_\phi + p_r\dot{r}] = 0. \end{aligned}$$

If we evaluate at the origin then:

$$P_{\phi\phi}^0 = -[\bar{f}_{\phi\phi\phi}^0 + (\bar{f}_{\phi r}^0 + \bar{f}_{\phi p}^0 p_r^0)\dot{r}](\bar{f}_{\phi p}^0)^{-1}.$$

Hence by conditions 2 and 4 above, we get:

$$p_{\phi\phi}^0 \neq 0,$$

which ends the proof. □

3.5.1 Hopf bifurcation

Hopf bifurcation (HB) does not yield any changes in the number of equilibrium points. Hopf bifurcations are characterized by an equilibrium manifold of an $(\epsilon = 0)$ along complex eigenvalues of the fast field at the equilibrium cross the imaginary axis transversally

and no other eigenvalue with zero real part and the stability of the critical point are lost through its interaction with a limit cycle. There exist two types of Hopf Bifurcation depending on the nature of the interaction with a limit cycle [46]:

- Subcritical HB: an unstable limit cycle, existing prior to the bifurcation, shrinks and eventually disappears as it coalesces with a stable equilibrium point at the bifurcation. After the bifurcation, the equilibrium point becomes unstable resulting in growing oscillations.
- Supercritical HB: a stable limit cycle is generated at the bifurcation, and a stable equilibrium point becomes unstable with increasing amplitude oscillations, which are eventually attracted by the stable limit cycle.

To study Hopf bifurcation we consider blow up method and equation

$$\begin{aligned}\frac{dx}{dt} &= \dot{x} = f(x, y, \epsilon), \\ \frac{dy}{dt} &= \dot{y} = \epsilon g(x, y, \epsilon),\end{aligned}$$

where $f : R^n \times R^m \times R \longrightarrow R^n$, $g : R^n \times R^m \times R \longrightarrow R^m$. In this case we can talk about vector fields around critical points. These vector fields can be got by considering eigenvalues given by characteristic polynomial:

$$L(\lambda) = \det(J - \lambda I),$$

where I is the identity matrix and J is the Jacobian matrix evaluated at critical point. If we constraint our work in three dimensional space, then we can write:

$$\det(J - \lambda I) = \det \begin{bmatrix} f_x - \lambda & f_y & f_\epsilon \\ g_x & g_y - \lambda & g_\epsilon \\ 0 & 0 & 0 - \lambda \end{bmatrix} = \lambda^3 + T\lambda^2 - D\lambda = 0, \quad (3.5.2)$$

where T is the trace of the matrix J and D is its determinate. The following theorem which is gives a sufficient conditions which for $(r_*, \phi_*, \epsilon_*)$ which are an equilibrium point in (3.1.2) to undergoe a Hopf bifurcation.

Theorem 7. *Consider singular perturbation system (3.1.7) if:*

1. $\bar{f}_\phi(r_*, \phi_*, p_*, \epsilon_*)\bar{g}_r(r_*, \phi_*, p_*, \epsilon_*) - \bar{f}_\phi(r_*, \phi_*, p_*, \epsilon_*)(\bar{g}_r(r_*, \phi_*, p_*, \epsilon_*)) > 0,$
2. $\bar{f}_\phi(r_*, \phi_*, p_*, \epsilon_*) + \bar{g}_\phi(r_*, \phi_*, p_*, \epsilon_*) = 0,$

then $(r_*, \phi_*, p_*, \epsilon_*)$ is a Hopf bifurcation point.

Proof. If we consider the Jacobian matrix for (3.1.2) at the origin point, then we get a trace $T(r_*, \phi_*, p_*, \epsilon_*)$, which equals zero by condition 2 above. And the determinate $D(r_*, \phi_*, p_*, \epsilon_*)$, which is greater than zero by condition 1. Hence (3.5.2) should have two pure imaginary roots. □

Chapter 4

Bifurcation Theory in Differential Algebraic Equations System By Blow up Method

4.1 Introduction

The dynamics of a large systems can be modeled by parameter dependent (DAEs) of the form :

$$\dot{x} = f(x, y, p), \tag{4.1.1}$$

$$0 = g(x, y, p).$$

Where $(x, y) \in R^n \times R^m$ and the state x is directly defined by f while the variables y are such the system that satisfies the constraints g . The parameter $p \in R^l$ is defined as a specific system configuration and the operating condition.

Standard bifurcation theorem deals with the study of the stability of ODEs that moves from equilibrium point to equilibrium as the parameters of the system are slowly changing . Let C be the set of equilibrium points of (4.1.1) where

$$C = \{(x, y, p, 0) \in R^2 \times R \times R, f(x, y, p, 0) = g(x, y, p, 0) = 0\}. \quad (4.1.2)$$

And define a function F by:

$$F(x, y, p, 0) = \begin{bmatrix} f(x, y, p, 0) \\ g(x, y, p, 0) \end{bmatrix}. \quad (4.1.3)$$

4.2 Directional Blow up in Differential Algebraic Equations

In order to simplify some of the calculations, we introduce another description of the blowing up construction . We will see in directional case how it depends on DAE system. Here we rescall different strategy and give theorems and an example to explain this case. In [11], the directional blow up is given on ODEs. Now we give the following definition for blow up on DAEs.

Definition 19. Let F be a vector field as defined in (4.1.3) with $F(0) = 0$. Define the map $\mu_i : R^2 \longrightarrow R^2$ by

$$\mu_i(x, y, 0) = (x, xy, 0),$$

where $i = 1, 2$. Then directional blow up of F is defined by (4.1.3):

$$\hat{F}^i(x, y, 0) = ((D\mu_i)^{-1} \circ F \circ \mu_i)(x, y, 0), \quad (4.2.1)$$

and the directional blow up of DAE (4.1.1) is:

$$\hat{f}^i(x, y, 0) = ((D\mu_i)^{-1} \circ f \circ \mu_i)(x, y, 0), \quad (4.2.2)$$

$$\hat{g}^i(x, y, 0) = 0 = ((D\mu_i)^{-1} \circ g \circ \mu_i)(x, y, 0). \quad (4.2.3)$$

If $J_l(F)(0) = 0$ for $l \in \{0, 1, \dots, k\}$ and $J_{k+1}(F)(0) \neq 0$, where J_l is jacobian matrix for F and J_{k+1} is jacobian matrix for $\bar{F}^i$ where

$$\bar{F}^i(x, y, 0) = \frac{1}{x} \hat{F}_i(x, y, 0), \quad (4.2.4)$$

which defines the rescaled directional blowup in direction x , and

$$\begin{aligned} \bar{f}^i(x, y, 0) &= \frac{1}{x} \hat{f}^i(x, y, 0) \\ \bar{g}^i(x, y, 0) &= 0 = \frac{1}{x} \hat{g}^i(x, y, 0). \end{aligned}$$

Example 5. Consider the DAE system

$$\dot{x} = x^2 - 2xy,$$

$$0 = y^2 - 2xy.$$

In the x -direction, one gets:

$$\bar{F}^1 = \frac{1}{x} \begin{bmatrix} 1 & 0 \\ -\frac{y}{x} & \frac{1}{x} \end{bmatrix} \begin{bmatrix} x^2 - 2x^2y \\ x^2y^2 - 2x^2y \end{bmatrix} = \begin{bmatrix} x(1 - 2y), \\ 3y(y - 1). \end{bmatrix}$$

In y -direction becomes

$$\bar{F}^2 = \frac{1}{y} \begin{bmatrix} \frac{1}{y} & -\frac{x}{y} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^2y^2 - 2xy \\ y^2 - 2xy^2 \end{bmatrix} = \begin{bmatrix} 3x(x - 1), \\ y(1 - 2x), \end{bmatrix}$$

The following lemma explain the relation between polar bowing up and directional blowing up.

Lemma 1. [29] *The directional and polar blow up methods are equivalent. Then a sufficiently small $V \in R^2$ is given. We can find $U \subset S^1 \times (-\frac{1}{2}, \infty)$ s.t. $\hat{F}$ and $\hat{F}^i$ are equivalent up to a diffeomorphism*

$$C : U \longrightarrow V.$$

given by $C(r, \phi) = (r \cos \phi, \tan \phi)$.

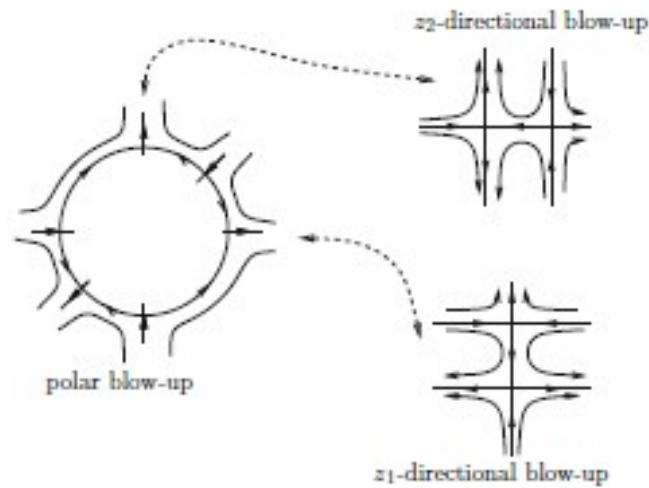


Figure 4.1: phase portrait to explain the equivalence between $\hat{F}$ and $\hat{F}^i$

In this chapter we apply directional blow up method to get some types of bifurcation.

4.3 Fold Bifurcation in DAEs

In this section, we extend bifurcation theory to a system of differential algebraic equation (4.1.1) which will be reduced by blowing up method. The reduced equation is given by (4.2.2) and (4.2.3). For this, we need the following assumptions:

1. $(0, 0, 0, 0)$ is an equilibrium point *i.e.* $\bar{f}^i(0, 0, 0, 0) = \bar{g}^i(0, 0, 0, 0)$.
2. $\bar{f}_x^i(0, 0, 0, 0) = \bar{g}_x^i(0, 0, 0, 0) = 0$.
3. $\bar{f}_{xx}^i(0, 0, 0, 0) \neq 0$.

The next lemma give guarantee for the origin point in (4.1.1) to be a bifurcation point.

Lemma 2. [26] Suppose that $\bar{f}_{xx}^i(0) \neq 0$ or $\bar{f}_{yy}^i(0, 0, 0, 0) \neq 0$ and assumption 2 above, then the origin is bifurcation point of (4.1.1).

The following theorem proves the origin point which is fold bifurcation by applying directional blow up on DAEs systems.

Theorem 8. Consider the DAE system (4.1.1) with the set of equilibrium points C . If the following conditions hold:

- 1 $\bar{g}_p^i(0, 0, 0, 0) \neq 0$ and,
- 2 $\bar{g}_{xx}^i(0, 0, 0, 0) \neq 0$,

where $\bar{g}^i$ as defined in (4.2.4). Then $(0, 0, 0, 0)$ is a saddle node (Fold) bifurcation point.

Proof. Lemma (2) guarantee that $(0, 0, 0, 0)$ is bifurcation point. By the above condition 1

$$\bar{F}_p^i \neq 0,$$

then by implicit function theorem, we have:

$$\bar{F}^i(x, y, p(x, y)) = 0. \quad (4.3.1)$$

Differentiate (4.3.1) w.r.t x , we get :

$$\bar{F}_x^i + (\bar{F}_y^i \dot{y} + \bar{F}_p^i(p_x + p_y \dot{y})) = 0.$$

Evaluating at the origin, we obtain:

$$\bar{F}_x^i(0, 0, 0, 0) + \bar{F}_p^i(0, 0, 0, 0)p_x(0, 0, 0, 0) = 0.$$

Now by applying condition one above and the assumption 2 ($\bar{g}_x^i(0, 0, 0, 0) = 0$), we obtain:

$$p_x(0, 0, 0, 0) = -\bar{g}_x^i(0, 0, 0, 0)\bar{g}_p^i(0, 0, 0, 0)^{-1} = 0$$

Now take the second derivative w.r.t x :

$$\begin{aligned} & \bar{F}_{xx}^i + \bar{F}_{xy}^i \dot{y} + \bar{F}_{xp}^i(p_x + p_y \dot{y}) + \bar{F}_y^i \dot{y}[\bar{F}_{yx} + \bar{F}_{yy}^i \dot{y} + \bar{F}_{yp}^i(p_x + p_y \dot{y}) + \bar{F}_p^i] \\ & [p_{xx} + p_{xy} \dot{y} + p_y \dot{\dot{y}} + (p_{yx} + p_{yy} \dot{y})\dot{y}] + [p_x + p_y \dot{y}][\bar{F}_{px}^i + \bar{F}_{py}^i \dot{y} + \bar{F}_{pp}^i(p_x + p_y \dot{y})] = 0. \end{aligned}$$

Evaluating at $(0, 0, 0, 0)$ and substituting $p_x(0, 0, 0, 0) = 0$ as we proved, implies:

$$\begin{aligned} & \bar{F}_{xx}^i(0, 0, 0, 0) + \bar{F}_y^i(0, 0, 0, 0)\dot{\dot{y}} + \bar{F}_p^i p_{xx}(0, 0, 0, 0) + \bar{F}_p^i p_y(0, 0, 0, 0)\dot{\dot{y}} = 0, \\ & \bar{g}_{xx}^i(0, 0, 0, 0) + (\bar{g}_y^i(0, 0, 0, 0) + \bar{g}_p^i p_y(0, 0, 0, 0))\dot{\dot{y}} + \bar{g}_p^i p_{xx}(0, 0, 0, 0) = 0, \end{aligned}$$

then by condition one above, we can write:

$$p_{xx}(0, 0, 0, 0) = -[\bar{g}_{xx}^i(0, 0, 0, 0) + (\bar{g}_y^i(0, 0, 0, 0) + \bar{g}_p^i p_y(0, 0, 0, 0))\dot{y} + ((\bar{g}_y^i(0, 0, 0, 0))(\bar{g}_p^i(0, 0, 0, 0))^{-1}.$$

Finally, with both conditions 1 and 2, it is certainly $p_{xx}(0, 0, 0, 0) \neq 0$, which ends the proof. $\square$

4.4 Transcritical Bifurcation in DAEs

In this section, we apply the conditions of this kind of bifurcation on directional blow up. We write the function $\bar{F}^i$ given in (4.2.4) where the functions U and V are defined as:

$$\bar{f}^i(x, y, p, 0) = xU(x, y, p, 0),$$

$$\bar{g}^i(x, y, p, 0) = xV(x, y, p, 0).$$

Since

$$\bar{F}^i(x, y, p, 0) = \begin{bmatrix} \bar{f}^i(x, y, p, 0) \\ \bar{g}^i(x, y, p, 0) \end{bmatrix} = \begin{bmatrix} xU(x, y, p, 0) \\ xV(x, y, p, 0) \end{bmatrix},$$

and $U(0, y, p, 0) = \frac{\partial \bar{f}^i}{\partial x}$ at $x = 0$, $V(0, y, p, 0) = \frac{\partial \bar{g}^i}{\partial x}$ at $x = 0$.

So $x = 0$ is a surface of critical points passing through the origin from side to side of $p = 0$. Hence, all what we need to do is finding another surface, in $U = 0$ or $V = 0$, passing through the origin from side to side of $p = 0$, (we need to prove $p_x(0, 0, 0, 0) \neq 0$.)

Theorem 9. *Consider the DAE system (4.1.1) with the surface of equilibrium point C . If*

$$1 \quad \bar{g}_p^i(0, 0, 0, 0) = 0,$$

2 $\bar{g}_{xp}^i(0, 0, 0, 0) \neq 0$, and

3 $\bar{g}_{xx}^i(0, 0, 0, 0) \neq 0$,

then $(0, 0, 0, 0)$ is a Transcritical bifurcation point.

Proof. Consider the function $V(x, y, p, 0) = g_x$. Since $\bar{g}_{xp}^i \neq 0$, then by definition of the function V :

$$V_p(x, y, p, 0) \neq 0.$$

And by implicit function theorem:

$$V(x, y, p(x, y), 0) = 0,$$

then we differentiate w.r.t x .

$$V_x(x, y, p) + V_y(x, y, p)\dot{y} + V_p(x, y, p, 0)(p_x + p_y(x, y)\dot{y}) = 0.$$

If we evaluate at the origin, and by conditions 2 and 3:

$$\bar{g}_{xx}^i(0, 0, 0, 0) + \bar{g}_{xp}^i(0, 0, 0, 0)p_x(0, 0, 0, 0) = 0$$

So

$$p_x(0, 0, 0, 0) \neq 0$$

which ends the proof. □

4.5 Pitchfork Bifurcation in DAEs

To get a pitchfork bifurcation at the origin, we need to find two surfaces of critical points passing through the origin. One is from side to side of $p = 0$ and the other should be

placed entirely in one side of $p = 0$, i.e. , $p_x(0, 0, 0, 0) = 0$ and $p_{xx}(0, 0, 0, 0) \neq 0$. The following lemma formulate the conditions of Pitchfork bifurcation to occur.

Theorem 10. *Consider the DAE system (4.1.1) with the surface of equilibrium point C . If*

$$\mathbf{1} \quad \bar{g}_p^i(0, 0, 0, 0) = 0,$$

$$\mathbf{2} \quad \bar{g}_{xp}^i(0, 0, 0, 0) \neq 0,$$

$$\mathbf{3} \quad \bar{g}_{xx}^i(0, 0, 0, 0) = 0,$$

$$\mathbf{4} \quad \bar{g}_{xxx}^i(0, 0, 0, 0) \neq 0,$$

then $(0, 0, 0, 0)$ is a pitchfork bifurcation point.

Proof. Lemma (2) guarantee $(0, 0, 0, 0)$ which is a bifurcation point. Consider the function $V(x, y, p, 0) = g_x$ (see section(4.4)). Since $\bar{g}_{xp}^i(0, 0, 0, 0) \neq 0$ then by definition of the function V :

$$V_p(x, y, p, 0) \neq 0.$$

Then by implicit function theorem :

$$V(x, y, p(x, y), 0) = 0, \tag{4.5.1}$$

which can be differentiated w.r.t x :

$$V_x(x, y, p, 0) + V_y(x, y, p, 0)\dot{y} + V_p(x, y, p, 0)(p_x(x, y, 0) + p_y(x, y, 0)\dot{y}) = 0.$$

If we evaluate at the origin we get:

$$\bar{g}_{xx}^i(0, 0, 0, 0) + \bar{g}_{xp}^i(0, 0, 0, 0)p_x(0, 0, 0, 0) = 0.$$

By conditions 2 and 3, we get:

$$p_x(0, 0, 0, 0) = 0.$$

It can be proved that $p_{xx}(0, 0, 0, 0) \neq 0$, to do so we differentiate (4.5.1) two times w.r.t x :

$$\begin{aligned} & V_{xx} + V_{xy}\dot{y} + V_{xp}(p_x + p_y\dot{x} + V_y\dot{y} + [V_{yx} + V_{yy}\dot{y} + V_{yp}(p_x + p_y\dot{y} + \\ & V_p[p_{xx} + p_{xy}\dot{y} + p_y\dot{y}' + (yp_x + p_{yy}\dot{y})\dot{y}] + [V_{px} + V_{py}\dot{y} + V_{pp}(p_x + p_y\dot{y})][p_x + p_y\dot{y}] = 0. \end{aligned}$$

If we evaluate at the origin then:

$$p_{xx}(0, 0, 0, 0) = -[\bar{g}_{xxx}^i(0, 0, 0, 0) + (\bar{g}_{xy}^i(0, 0, 0, 0) + \bar{g}_{xp}^i p_y(0, 0, 0, 0))\dot{y}'](\bar{g}_{xp}^i)(0, 0, 0, 0)^{-1}.$$

Hence by conditions 2 and 4 above, we get:

$$p_{xx}(0, 0, 0, 0) \neq 0,$$

which ends the proof. □

Example 6. Consider the DAE system

$$\dot{x} = -xy^3,$$

$$0 = py - y^3.$$

By applying directional blow up we get:

$$\dot{x} = -xy - pxy + xy,$$

$$0 = py - y^3.$$

There are two curves of equilibrium points passing through the origin, one is $(0, 0)$ and the other is $(0, \mp\sqrt{p})$. The first one exists on both sides of $p = 0$ and the other exists on one side only. $(0, \mp\sqrt{p})$ should satisfy $p_x^0 = 0$ and $p_{xx}^0 \neq 0$.

To understand pitchfork bifurcation by blow up method we made figures(4.1)(4.2).

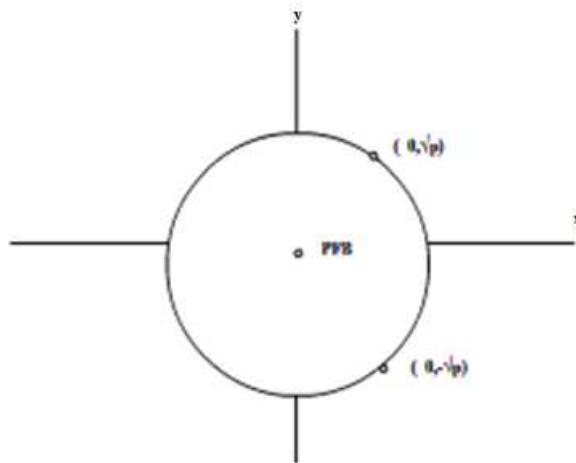


Figure 4.2: a: Pitchfork bifurcation by blow up method

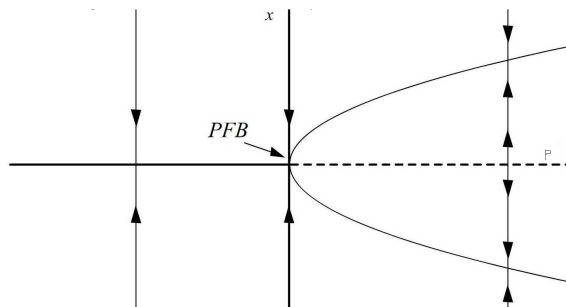


Figure 4.3: b: Pitchfork bifurcation for F

4.6 Hopf bifurcation in DAEs

As it is known this type of bifurcation occurs at a point where the system has simple pair of purely imaginary eigenvalues ($\lambda = iw$) and no other eigenvalue with zero real

part. If we constraint our work in three dimensional space, then we can write:

$$\det \begin{bmatrix} f_x - \lambda & f_y \\ g_x & g_y - \lambda \end{bmatrix} = \lambda^2 - T\lambda + D = 0, \quad (4.6.1)$$

where T is the trace of the matrix J and D is its determinate. Then the two eigenvalues of this system are given by:

$$\lambda_{1,2} = \frac{T \mp \sqrt{T^2 - 4D}}{2} \quad (4.6.2)$$

The following theorems give sufficient conditions for $(x_*, y_*, 0)$ which are equilibrium points in (4.1.1) to undergo a Hopf bifurcation.

Theorem 11. *Consider the index one DAE (4.1.1) if:*

- 1 $\bar{f}_x^i(0, 0, 0, 0) + \bar{g}_y^i(0, 0, 0, 0) = 0.$
- 2 $\bar{f}_x^i(0, 0, 0, 0)\bar{g}_y^i(0, 0, 0, 0) - \bar{f}_y^i(0, 0, 0, 0)\bar{g}_x^i(0, 0, 0, 0) > 0$

then $(0, 0, 0, 0)$ is a Hopf bifurcation point.

Proof. From condition one above, we get a trace:

$$T(0, 0, 0, 0) = \bar{f}_x^i(0, 0, 0, 0) + \bar{g}_y^i(0, 0, 0, 0),$$

which equals zero. The determinate:

$$D(0, 0, 0, 0) = \bar{f}_x^i(0, 0, 0, 0)\bar{g}_y^i(0, 0, 0, 0) - \bar{f}_y^i(0, 0, 0, 0)\bar{g}_x^i(0, 0, 0, 0),$$

which is greater than zero by condition two. The Substitute is in (4.6.2) the result is hold. Hence (4.1.1) should have two pure imaginary roots. $\square$

Chapter 5

Conclusion and Future Work

Conclusion

In this study, It is shown that the differential algebraic equation (*DAEs*) can be studied through the singular perturbed system by blow up method.

The classical theorems of bifurcation can be applied to (*DAEs*). The known kinds of bifurcation:

fold, transcritical, pitchfork and hopf are investigated for DAEs by using directional blow up method and Implicit Function Theorem.

Also the case $0 < \epsilon \ll 1$ is carefully dealt with so as to show that bifurcation which occurs in singularity perturbed ODEs by using polar blow up method and Implicit Function Theorem.

Future Work

This study can be generalized by looking at the following subjects on the singularity perturbed ODEs:

- Applying singular induced bifurcation on singularity perturbed ODEs by using blow up method.
- Studying the periodic orbits of singularity perturbed ODEs and then applying the global bifurcation kinds such as (homoclinic, heteroclinic, period doubling and blow-sky bifurcation) by depending on Tilkinov Theory for limit cycles of the Hamiltonian system,
- Applying double bifurcation theory on singularity perturbed ODEs of order two with changing the order of time scale,
- Try to the reduced model of singularity perturbed ODEs by using the integration methods and using the resulting equation in control system by using the application of engineering, physics processes.

List of Publications

- (1) H. K. Mnahi, K. H. Yasir. Bifurcation Theory in Singularity Parameterized ODEs, vector field, Journal of advances in mathematics. 2017, 12,2347-1921.
- (2) H.K.Mnahi, K. H. Yasir. Bifurcation by Application Blow up method to DAE submitted to Journal of Zankoy Sulaimani-Part A(JZS-A).

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